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Analysis of Time-Space Translations in Quantum Fields

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I discuss relativistic quantum fields whose time-space translations are realized in indefinite unitary groups. Such indefinite metric fields describe interactions, e.g., the Coulomb interaction, which cannot be parametrized completely by particles. They cannot be expanded with time-space translation eigenvectors; the representations of the translations involved are triangularizable, but not diagonalizable. To project on a positive-definite subspace for a probability interpretation, the vanishing of the nilpotent part in the time-space translations realization is required. A trivial Becchi–Rouet–Stora charge (gauge invariance) for the asymptotics in quantum gauge theories can be interpreted as one special case of this general principle—the asymptotic projection to the eigenvectors of the time-space translations.

NOTATIONAL PRELIMINARIES

Throughout this paper definite units for the—apparently—threefold dimensional graduation in physics are assumed: $\hbar$ (Planck's action unit), c (Einstein's velocity unit), and an unspecified mass unit μ_0. With such a basis all masses and energy-momenta come as real numbers.

Relativistic fields are symbolized with boldface letters, e.g., $\mathbf{\Phi}(x)$, $\mathbf{Z}(x)$, $\mathbf{l}(x)$, $\mathbf{b}(x)$, etc., their harmonic components with Roman letters, e.g., e, U, a, b, etc.

For Lie groups, $\mathbf{U}(n_+, n_-)$ and $\mathbf{SU}(n_+, n_-)$ with $n_+ + n_- = n$ stand for the unitary and special unitary groups. $\mathbf{O}(n_+, n_-)$ and $\mathbf{SO}(n_+, n_-)$ denote the real orthogonal groups, $\mathbf{SO}^+(1, n)$ the orthochronous groups. The notations $\mathbf{GL}(\mathbb{C}^n)$, $\mathbf{SL}(\mathbb{C}^n)$ and $\mathbf{GL}(\mathbb{R}^n)$, $\mathbf{SL}(\mathbb{R}^n)$ are used for the complex and real general n^2-dimensional and special $(n^2 - 1)$-dimensional groups. If $\mathbf{GL}(\mathbb{C}^n)$

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and $\mathbf{SL}(\mathbb{C}^n)$ are considered as real Lie groups with dimension $2n^2$ and $2(n^2 - 1)$, respectively, they are denoted with a subindex R, e.g., $\mathbf{GL}(\mathbb{C}^n)_\mathbb{R}$ and $\mathbf{SL}(\mathbb{C}^n)_\mathbb{R}$.

For groups realized in endomorphisms (matrix groups) a more individual notation proves useful. The $\mathbf{U}(1)$ isomorphic phase group for a d-dimensional complex space is written as $\mathbf{U}(1_d)$. If $\mathbf{U}(1)$ is realized in $\mathbf{SU}(2)$ by

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

the notation $\mathbf{U}(1)_3$ will be used, in $\mathbf{SU}(2d)$ the notation $\mathbf{U}(1_d)_3$. If $\mathbf{U}(1)$ comes in $\mathbf{U}(2)$ as

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

it will be called $\mathbf{U}(1)_+$ and correspondingly $\mathbf{U}(1)_-$ and $\mathbf{U}(1_d)_\pm$. Analogous notations will be used also for other groups, e.g., $\mathbf{SL}(\mathbb{C}_2^n)_\mathbb{R}$ for

$$\begin{pmatrix} \mathbf{SL}(\mathbb{C}^n)_\mathbb{R} & 0 \\ 0 & \mathbf{SL}(\mathbb{C}^n)_\mathbb{R} \end{pmatrix}$$

The groups $\mathbf{U}(n_+, n_-) = \mathbf{U}(1_n) \circ \mathbf{SU}(n_+, n_-)$ are the product of two normal subgroups, the phase group and the special group. Because of the cyclic group $\mathbf{I}_n = \{z \in \mathbb{C} \mid z^n = 1\}$ as intersection $\mathbf{U}(1_n) \cap \mathbf{SU}(n_+, n_-) \cong \mathbf{I}_n$, the product is not direct for $n \geq 2$. The group $\mathbf{GL}(\mathbb{C}^n)_\mathbb{R} = \mathbf{D}(1_n) \times \mathbf{UL}(\mathbb{C}^n)_\mathbb{R}$ is the direct product of the normal subgroups $\mathbf{D}(1_n)$ (dilatations) and $\mathbf{UL}(\mathbb{C}^n)_\mathbb{R} = \mathbf{U}(1_n) \circ \mathbf{SL}(\mathbb{C}^n)_\mathbb{R}$, the latter being the product of the phase group and the special linear group, not direct for $n \geq 2$.

The Lie algebras for the groups will be denoted with corresponding lowercase letters, e.g., $\mathbf{u}(1)$ for $\mathbf{U}(1)$, $\mathbf{sl}(\mathbb{C}^n)_\mathbb{R}$ for $\mathbf{SL}(\mathbb{C}^n)_\mathbb{R}$, etc.

INTRODUCTION

Wigner's (1939) particle classification relies on the harmonic analysis of the Poincaré group in terms of $\mathbf{U}(1)$-characters for the time-space translations x, i.e., positive unitary representations $e^{ixq} \in \mathbf{U}(1)$ with real energies $q_0 = (m^2 + \vec{q}^2)^{1/2}$. In the course of such an analysis the semidirect product Poincaré group $\mathbf{SO}^+(1, 3) \times_s \mathbb{M}$ with the orthochronous Lorentz group $\mathbf{SO}^+(1, 3)$ and the Minkowski time-space translations $\mathbb{M} \cong \mathbb{R}^4$ as action group for fields is reduced to an action group for particles, given by a direct product group $\mathbf{SO}(n) \times \mathbb{T}$ with a homogeneous compact group $\mathbf{SO}(n) \subset \mathbf{SO}^+(1, 3)$, $n = 3, 2$, as the stability group for a 1-dimensional time translation group $\mathbb{T} \cong \mathbb{R}$. The cases for massive and massless particles have to be distinguished.

For a particle with nontrivial mass $m^2 = q^2 > 0$, the stability group $\mathbf{SO}(3)$ (spin) describes the rotation degrees of freedom of the rest frames which are characterized by the energy-momenta $\underline{q}(m) = (m, 0, 0, 0)$. An associated Sylvester decomposition splits the Minkowski space $\mathbb{M} \cong \mathbb{T} \oplus \mathbb{S}^3$ into time and space translations $\mathbb{S}^3 \cong \mathbb{R}^3$.

Massless particles with lightlike energy-momenta $q^2 = 0$, $q \neq 0$, have no rest systems. In this case, the Minkowski translations have to be Witt-decomposed $\mathbb{M} \cong \mathbb{L}_+ \oplus \mathbb{S}^2 \oplus \mathbb{L}_-$ into two 1-dimensional lightlike translation spaces $\mathbb{L}_\pm \cong \mathbb{R}$ and 2-dimensional space translations $\mathbb{S}^2 \cong \mathbb{R}^2$. The stability group of those time-space translation frames which are determined by two independent lightlike vectors $\underline{q}(\mu_\pm) = \mu_\pm(1, 0, 0, \pm 1)$ or, equivalently, by one nontrivial timelike and one spacelike vector $\mathbb{L}_+ \oplus \mathbb{L}_- \cong \mathbb{T} \oplus \mathbb{S}^1$ with $\underline{q}(\mu) = (\mu, 0, 0, 0)$ and $\underline{q}(\kappa) = (0, 0, 0, \kappa)$, is the circularity (helicity, polarization) group $\mathbf{SO}(2)$.

Collecting both cases, there arises the following scheme of Minkowski space decompositions with their particle's relevant stability groups:

$$\mathbf{SO}^+(1,3) \text{ for } \mathbb{M} \begin{array}{l} \supset \mathbf{SO}(3) \text{ for } \mathbb{T} \oplus \mathbb{S}^3 \quad (m^2 > 0) \\ \supset \mathbf{SO}(2) \text{ for } \mathbb{L}_+ \oplus \mathbb{S}^2 \oplus \mathbb{L}_- \quad (m^2 = 0) \end{array}$$

In the framework of quantum theory, the time-space translations and the Lorentz group, both real Lie groups, are realized on complex spaces, i.e., they come as subgroups of unitary groups, not necessarily positive unitary.

The Lorentz group comes in the group $\mathbf{SL}(\mathbb{C}^2)$, considered as a real 6-dimensional Lie group and denoted by $\mathbf{SL}(\mathbb{C}^2)_\mathbb{R}$, with the isomorphy $\mathbf{SO}^+(1, 3) \cong \mathbf{SL}(\mathbb{C}^2)_\mathbb{R}/\mathbb{I}_2$, where $\mathbb{I}_2 = \{\pm 1\}$ is the sign group (real phases).

For Weyl spinor fields, the Lorentz symmetry $\mathbf{SL}(\mathbb{C}^2)_\mathbb{R}$ is represented as subgroup of the indefinite unitary group $\mathbf{U}(2, 2)$, where it is accompanied by a phase group $\mathbf{U}(1)$. Starting from the phase Lorentz group

$$\mathbf{UL}(\mathbb{C}^2)_\mathbb{R} = \{\lambda \in \mathbf{GL}(\mathbb{C}^2)_\mathbb{R} \,|\, |\det \lambda| = 1\} = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SL}(\mathbb{C}^2)_\mathbb{R}$$

the orthochronous Lorentz group is the manifold of the phase $\mathbf{U}(\mathbf{1}_2)$-orbits, i.e., $\mathbf{UL}(\mathbb{C}^2)_\mathbb{R}/\mathbf{U}(\mathbf{1}_2) \cong \mathbf{SO}^+(1, 3)$. Massive Majorana and massless Weyl particles are characterized by the subgroups

$$\mathbf{SL}(\mathbb{C}^2)_\mathbb{R} \subset \mathbf{UL}(\mathbb{C}^2)_\mathbb{R} \begin{array}{l} \supset \left\{ \begin{array}{c} \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SU}(2) \\ \text{Majorana particles} \\ (m^2 > 0) \end{array} \right. \\ \supset \left\{ \begin{array}{c} \mathbf{U}(1) \times \mathbf{U}(1) \\ \text{Weyl particles} \\ (m^2 = 0) \end{array} \right. \end{array}$$

The stability group for Weyl particles is a $\mathbf{U}(1)$-circularity (polarization) with $\mathbf{U}(1) \cong \mathbf{SO}(2)$; for Majorana particles one has spin $\mathbf{SU}(2)$ with $\mathbf{SU}(2)/\mathbb{I}_2$

$\cong$ **SO**(3). The additional **U**(1) group realizes the time-space translations (Section 1).

The stability group for Dirac particles is spin **SU**(2) and, in addition, an internal charge group **U**(1) which arises because of the twofold left–right-handed Lorentz group representation involved,

$$\mathbf{SL}(\mathbb{C}^2)_{\mathbb{R}} \subset \mathbf{U}(1) \times \mathbf{UL}(\mathbb{C}^2)_{\mathbb{R}} \supset \underset{\substack{\text{Dirac particles}\\ (m^2 > 0)}}{\mathbf{U}(1) \times \mathbf{U}(1_2) \circ \mathbf{SU}(2)}$$

For vector fields, the Lorentz group $\mathbf{SO}^+(1, 3)$ is represented as subgroup of the indefinite unitary group **U**(1, 3), compatible with the Lorentz "metric" (−1, 1, 1, 1). The arising field types are given in the scheme

$$\mathbf{SO}^+(1, 3) \subset \mathbf{U}(1, 3) \begin{cases} \supset \begin{cases} \mathbf{U}(1) \circ \mathbf{U}(3) \supset \mathbf{U}(1) \times \mathbf{SO}(3) \\ \text{Sylvester particles} \\ (m^2 > 0) \end{cases} \\ \supset \begin{cases} \mathbf{U}(1, 1) \circ \mathbf{U}(2) \supset \mathbf{U}(1) \times \mathbf{SO}(2) \\ \text{Maxwell–Witt fields} \\ (m^2 = 0) \end{cases} \end{cases}$$

In the case of a Witt decomposition the indefinite Lorentz "metric" gives rise to the indefinite unitary subgroup **U**(1, 1) containing the time translation representations for the nonparticle contributions of the Maxwell–Witt fields (Saller *et al.*, 1995) (Section 2).

In general, positive and indefinite unitary groups realizing time-space translations will be called modality groups. They characterize the conjugations and inner products involved and, therewith, the probability interpretation of the theory. The symmetry group of a relativistic field dynamics, e.g., $\mathbf{SO}^+(1, 3)$ or $\mathbf{SL}(\mathbb{C}^2)_{\mathbb{R}}$, should be distinguished[2] from the unitary modality group, e.g., **U**(1, 3) or **U**(2, 2).

Representations of the time-space translations in the positive-definite modality group **U**(1) are used for Wigner-classified particles. The corresponding fields have an analysis in terms of translation eigenvectors. Fields with translation representations in indefinite modality groups, e.g., the nonparticle Coulomb degree of freedom in massless gauge fields, Fadeev–Popov fields, etc. (Section 2), have no full particle analysis. The harmonic analysis cannot be performed with translation eigenvectors only. The mathematical structures involved, especially the connection between translations representation and metrical structure, are sketched in Section 3.

[2] An analogous situation is familiar from "dynamical symmetries," e.g. **U**(2, 2) for the nonrelativistic hydrogen atom containing the symmetries **SU**(2) × **SU**(2) for the bound states and $\mathbf{SL}(\mathbb{C}^2)_{\mathbb{R}}$ for the scattering states.

The main problem using fields which describe interactions without asymptotic particles is the unitarization, i.e., the establishment of a projection condition, compatible with the dynamics, to a state space with a positive inner product. It is shown in Section 4, how the projection to translation eigenstates coincides with the projection of the full algebra of fields to a subalgebra with positive inner product. In the case of Maxwell–Witt fields, the projection to time-space translation eigenstates coincides with the familiar gauge invariance condition (Becchi–Rouet–Stora invariance [Becchi *et al.*, 1976]) for quantum gauge fields.

1. PARTICLE FIELDS AND POSITIVE METRIC

In this section, positive metric quantum fields are considered which can be expanded with particles only, i.e., with translation eigenstates. To prepare the discussion of the less familiar fields with indefinite metric in Section 2, the representation properties of Dirac, Weyl, Majorana, and Sylvester particles are considered in some detail with respect to the represented action groups.

For a relativistic particle field $\Phi(x|m)$ which is symmetric with respect to a conjugation $*$ and which allows an analysis of the time-space translation properties with a Dirac measure for mass $m \geq 0$

$$\Phi_{\pm}(x|m) = \int \frac{d^4q\, e^{ixq}}{(2\pi)^3} \begin{pmatrix} 1 \\ -i\epsilon(q_0) \end{pmatrix} \delta(m^2 - q^2) e(q) = \Phi_{\pm}(x|m)^* \quad (1.1)$$

the energy-momentum reflected harmonic components $e(\pm q)$ are related to each other by the conjugation $*$,

$$\Phi_{\pm}(x|m) = \int \frac{d^3q}{(2\pi)^3 q_0} \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{e^{ixq}e(q) \pm e^{-ixq}e(q)^*}{2}\Bigg|_{q_0=(m^2+\vec{q}^2)^{1/2}} \quad (1.2)$$

$$e(q) = e(-q)^*$$

The real 4-dimensional additive group of the time-space translations $\mathbf{M} \cong \mathbf{R}^4$ is realized for particle fields in the real 1-dimensional compact unitary group $\mathbf{U}(1)$ with the energy-momenta q, $q^2 = m^2$, as eigenvalues

$$D_1(\cdot|q)\colon\quad \mathbf{M} \to \mathbf{U}(1), \quad \begin{cases} D_1(x|q) = e^{ixq} = D_1(-x|q)^\star \\ \partial^j|_{x=0} D_1(x|q) = iq^j \end{cases} \quad (1.3)$$

Because of the positive-definite modality group $\mathbf{U}(1)$ with conjugation $\star$, particle fields have a probability interpretation. The representation $D_1(x|q)$ of the time-space translations in $\mathbf{U}(1)$ is irreducible and not faithful.

The relation between the U(1)-conjugation ⋆ for the represented translations and the field conjugation * above has to take care of the spin properties involved.

1.1. Sylvester Particles

Sylvester particles will be defined as Bose particles with nontrivial mass $M > 0$ and stability group **SO**(3); they carry integer spin $s = 0, 1, \ldots$ representations.

For faithful representations of the Lorentz group $\mathbf{SO}^+(1, 3)$ with stability group **SO**(3), the defining representation can be exemplified by a massive vector field without internal charge degrees of freedom, e.g., the free neutral weak boson field $\mathbf{Z}^k$ of the standard model with mass $M > 0$. With a rest system determined up to space rotations, the time-space translation analysis for $\mathbf{Z}^k$ and its canonical partner $\mathbf{G}^{kj}$ read

$$\mathbf{Z}(x)^k = \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{M}\, \Lambda\left(\frac{q}{M}\right)^k_a \frac{e^{ixq}U(\vec{q})^a + e^{-ixq}\delta^{ab}U(\vec{q})^\star_b}{\sqrt{2}}$$

$$i\mathbf{G}(x)^{kj} = \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{M}\, \Lambda\left(\frac{q}{M}\right)^l_0 \epsilon^{kj}_{lr} \Lambda\left(\frac{q}{M}\right)^r_a \frac{e^{ixq}U(\vec{q})^a - e^{-ixq}\delta^{ab}U(\vec{q})^\star_b}{\sqrt{2}} \tag{1.4}$$

$$\epsilon^{kj}_{lr} = \delta^k_l \delta^j_r - \delta^j_l \delta^k_r$$

The boosts $\Lambda(q/M)$ with $q^2 = M^2$ transmutate from Lorentz vector fields to spinning particles, i.e., from $\mathbf{SO}^+(1, 3)$ to **SO**(3) representations with three spin directions $a = 1, 2, 3$,

$$\Lambda\left(\frac{q}{M}\right)^k_{0,a} \cong \frac{1}{M}\begin{pmatrix} q_0 & \vec{q} \\ \vec{q} & \mathbf{1}_3 M + \dfrac{\vec{q} \otimes \vec{q}}{q_0 + M} \end{pmatrix}, \qquad \Lambda(1, 0, 0, 0) = \mathbf{1}_4 \tag{1.5}$$

Those transmutators are representatives for the classes of the real 3-dimensional Sylvester manifold $\mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$.

The free field dynamics is illustrated by the classical $\mathbf{SO}^+(1, 3)$-invariant Lagrangian

$$\mathcal{L}(\mathbf{Z}, \mathbf{G}) = \mathbf{G}^{jk} \frac{\partial_j \mathbf{Z}_k - \partial_k \mathbf{Z}_j}{2} - \mathcal{I}(\mathbf{Z}, \mathbf{G}) \tag{1.6}$$

$$\mathcal{I}(\mathbf{Z}, \mathbf{G}) = -M\left(\frac{\mathbf{G}^{jk}\mathbf{G}_{jk}}{4} + \frac{\mathbf{Z}^j\mathbf{Z}_j}{2}\right)$$

With the representation of the Lorentz group in a unitary group $\mathbf{SO}^+(1, 3) \subset \mathbf{U}(1, 3)$, the stability spin group comes with a $\mathbf{U}(\mathbf{1}_3)$-conjugation, $\mathbf{U}(\mathbf{1}_3)$

$\times$ **SO**(3) $\subset$ **U**(1, 3). The positive-definite modality group $\mathbf{U}(1_3) \cong \mathbf{U}(1)$ represents the time-space translations. Its conjugation exchanges Z-creation operators $\mathrm{U}(\vec{q})^a$ with Z-annihilation operators $\mathrm{U}(\vec{q})_a^\star$,

$$\left.\begin{array}{r}\text{conjugation } \star \\ \text{for modality group } \mathbf{U}(1_3)\end{array}\right\} \mathrm{U}(\vec{q})^a \leftrightarrow \delta^{ab}\mathrm{U}(\vec{q})_b^\star \tag{1.7}$$

Lorentz vector fields are symmetric with respect to the conjugation $\star$, i.e., $\mathbf{Z} = \mathbf{Z}^\star$, $\mathbf{G} = \mathbf{G}^\star$.

The quantization and Fock-space positive inner product

$$\begin{aligned}[\mathrm{U}(\vec{p})_a^\star, \mathrm{U}(\vec{q})^b] &= \delta_a^b(2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \\ \langle\{\mathrm{U}(\vec{p})_a^\star, \mathrm{U}(\vec{q})^b\}\rangle &= \delta_a^b(2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) = \langle \mathrm{U}(\vec{p})_a^\star \mathrm{U}(\vec{q})^b\rangle\end{aligned} \tag{1.8}$$

lead to the field commutators and Fock values of the anticommutators, e.g.,

$$\begin{aligned}\begin{pmatrix}[\mathbf{Z}(y)^k, \mathbf{Z}(x)^j] \\ \langle\{\mathbf{Z}(y)^k, \mathbf{Z}(x)^j\}\rangle\end{pmatrix} &= -\left(\eta^{kj} + \frac{\partial^k\partial^j}{M^2}\right)\begin{pmatrix} i\mathbf{s}(x - y|M) \\ \mathbf{C}(x - y|M)\end{pmatrix} \\ &= \int \frac{d^3q\, e^{-i(\vec{x}-\vec{y})\vec{q}}}{(2\pi)^3 q_0} M\Lambda\left(\frac{q}{M}\right)_a^k \delta^{ab}\begin{pmatrix} i\sin(x_0 - y_0)q_0 \\ \cos(x_0 - y_0)q_0\end{pmatrix}\Lambda\left(\frac{q}{M}\right)_b^j\end{aligned} \tag{1.9}$$

with the quantization distribution **s** and the expectation function **C**

$$\begin{pmatrix}\mathbf{C}(x|m) \\ \mathbf{s}(x|m)\end{pmatrix} = \int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\begin{pmatrix} 1 \\ -i\epsilon(q_0)\end{pmatrix}\delta(m^2 - q^2) = \int \frac{d^3q\, e^{-i\vec{x}\vec{q}}}{(2\pi)^3 q_0}\begin{pmatrix}\cos x_0 q_0 \\ \sin x_0 q_0\end{pmatrix} \tag{1.10}$$

The modality group **U**(1), generated by $iI(\mathrm{U})$, is compatible with the stability group **SO**(3), generated by $i\vec{S}(\mathrm{U})$,

$$\begin{aligned} I(\mathrm{U}) &= \int \frac{d^3q}{(2\pi)^3 q_0}\frac{\{\mathrm{U}(\vec{q})^a, \mathrm{U}(\vec{q})_a^\star\}}{2} = I(\mathrm{U})^\star \\ S(\mathrm{U})^a &= \int \frac{d^3q}{(2\pi)^3 q_0} i\epsilon^{abc}\frac{\{\mathrm{U}(\vec{q})^b, \mathrm{U}(\vec{q})_c^\star\}}{2} = S(\mathrm{U})^{a\star} \\ [I(\mathrm{U}), \vec{S}(\mathrm{U})] &= 0 \end{aligned} \tag{1.11}$$

1.2. Dirac Particles

Dirac particles will be defined as Fermi particles with nontrivial mass $m > 0$ and stability group **U**(2); they feel half-integer spin **SU**(2) representations $s = 1/2$ and a nontrivial internal charge group **U**(1).

Faithful representations of the phase Lorentz group $\mathbf{UL}(\mathbf{C}^2)_\mathbf{R} = \mathbf{U}(1_2) \circ \mathbf{SL}(\mathbf{C}^2)_\mathbf{R}$ with stability group $\mathbf{U}(2)$ are exemplified by massive Dirac fields $\boldsymbol{\Psi} = (\mathbf{l}, \mathbf{r})$. They carry a decomposable phase Lorentz group representation with irreducible left- and right-handed Weyl contributions, illustrated by the free electron field of the standard model with mass $m > 0$. The time-space translations analysis for left- and right-handed contributions

$$
\begin{aligned}
\mathbf{l}(x)^A &= \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{m}\, \lambda\left(\frac{q}{m}\right)^A_\alpha \frac{e^{ixq}\mathrm{u}(\vec{q})^\alpha + e^{-ixq}\mathrm{a}(\vec{q})^{\star\alpha}}{\sqrt{2}} \\
i\mathbf{r}(x)^{\dot{A}} &= i \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{m}\, \hat{\lambda}\left(\frac{q}{m}\right)^{\dot{A}}_\alpha \frac{e^{ixq}\mathrm{u}(\vec{q})^\alpha - e^{-ixq}\mathrm{a}(\vec{q})^{\star\alpha}}{\sqrt{2}} \\
\mathbf{l}(x)^\star_{\dot{A}} &= \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{m}\, \lambda\left(\frac{q}{m}\right)^{\star\alpha}_{\dot{A}} \frac{e^{ixq}\mathrm{a}(\vec{q})_\alpha + e^{-ixq}\mathrm{u}(\vec{q})^\star_\alpha}{\sqrt{2}} \\
i\mathbf{r}(x)^\star_A &= -i \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{m}\, \lambda\left(\frac{q}{m}\right)^{-1\alpha}_A \frac{e^{ixq}\mathrm{a}(\vec{q})_\alpha - e^{-ixq}\mathrm{u}(\vec{q})^\star_\alpha}{\sqrt{2}}
\end{aligned}
\tag{1.12}
$$

involve electron and positron operators for creation $\mathrm{u}(\vec{q})$, $\mathrm{a}(\vec{q})$ and annihilation $\mathrm{u}(\vec{q})^\star$, $\mathrm{a}(\vec{q})^\star$.

The Weyl-represented boosts $\lambda(q/m)$ with $q^2 = m^2$ transmutate from spinor fields to particles, i.e., from $\mathbf{SL}(\mathbf{C}^2)_\mathbf{R}$ to $\mathbf{SU}(2)$ representations with two spin directions $\alpha = 1, 2$,

$$
\begin{aligned}
&\lambda\left(\frac{q}{m}\right) = \sqrt{\frac{q_0 + m}{2m}}\left(\mathbf{1}_2 + \frac{\vec{\sigma}\vec{q}}{q_0 + m}\right), \\
&\hat{\lambda}\left(\frac{q}{m}\right) = \lambda\left(\frac{q}{m}\right)^{\star-1} = \sqrt{\frac{q_0 + m}{2m}}\left(\mathbf{1}_2 - \frac{\vec{\sigma}\vec{q}}{q_0 + m}\right) \\
&\lambda(1, 0, 0, 0) = \mathbf{1}_2 = \hat{\lambda}(1, 0, 0, 0) \\
&\Lambda\left(\frac{q}{m}\right)^k_j = \frac{1}{2}\,\mathrm{tr}\, \lambda\left(\frac{q}{m}\right)\rho^k \lambda\left(\frac{q}{m}\right)^\star \check{\rho}_j, \qquad \lambda\left(\frac{q}{m}\right)^A_\alpha \lambda\left(\frac{q}{m}\right)^{\star\alpha}_{\dot{A}} = \frac{(\rho_k)^A_{\dot{A}} q^k}{m} \\
&\text{Weyl matrices:}\quad \check{\rho}_k = (\mathbf{1}_2, \vec{\sigma}), \quad \rho_k = (\mathbf{1}_2, -\vec{\sigma})
\end{aligned}
\tag{1.13}
$$

A classical $\mathbf{UL}(\mathbf{C}^2)_\mathbf{R}$-invariant Lagrangian reads

$$
\begin{aligned}
\mathscr{L}(\mathbf{l}, \mathbf{r}) &= i\mathbf{l}\check{\rho}_k \partial^k \mathbf{l}^\star + i\mathbf{r}\rho_k \partial^k \mathbf{r}^\star - \mathscr{I}(\mathbf{l}, \mathbf{r}) \\
\mathscr{I}(\mathbf{l}, \mathbf{r}) &= m(\mathbf{l}^A \mathbf{r}^\star_A + \mathbf{r}^{\dot{A}} \mathbf{l}^\star_{\dot{A}})
\end{aligned}
\tag{1.14}
$$

The quantization connects dual pairs

$$\{u(\vec{p})^\star_\alpha, u(\vec{q})^\beta\} = \{a(\vec{p})_\alpha, a(\vec{q})^{\star\beta}\} = \delta^\beta_\alpha(2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \tag{1.15}$$

The stability group conjugation

$$\left.\begin{matrix}\text{conjugation } \star \\ \text{for stability group } \mathbf{U}(\mathbf{1}_4) \circ \mathbf{SU}(\mathbf{2}_2)\end{matrix}\right\} \left\{\begin{matrix} u(\vec{q})^\alpha \leftrightarrow \delta^{\alpha\beta} u(\vec{q})^\star_\beta \\ a(\vec{q})^{\star\alpha} \leftrightarrow \delta^{\alpha\beta} a(\vec{q})_\beta \end{matrix}\right. \tag{1.16}$$

exchanges creation and annihilation operators.

The phase group $\mathbf{U}(\mathbf{1}_4) \cong \mathbf{U}(1)$, e.g., the electromagnetic charge group for electrons and positrons, is generated by $iI(u, a^\star)$

$$\begin{aligned} I(u, a^\star) &= I(u) + I(a^\star) \\ &= \int \frac{d^3q}{(2\pi)^3 q_0} \frac{[u(\vec{q})^\alpha, u(\vec{q})^\star_\alpha] + [a(\vec{q})^{\star\alpha}, a(\vec{q})_\alpha]}{2} \\ &= I(u, a^\star)^\star \end{aligned} \tag{1.17}$$

and the spin group $\mathbf{SU}(\mathbf{2}_2) \cong \mathbf{SU}(2)$ by $i\vec{S}(u, a^\star)$

$$\begin{aligned} \vec{S}(u, a^\star) &= \vec{S}(u) + \vec{S}(a^\star) \\ &= \int \frac{d^3q}{(2\pi)^3 q_0} \vec{\sigma}^\beta_\alpha \frac{[u(\vec{q})^\alpha, u(\vec{q})^\star_\beta] + [a(\vec{q})^{\star\alpha}, a(\vec{q})_\beta]}{2} \\ &= \vec{S}(u, a^\star)^\star \end{aligned} \tag{1.18}$$

The group $\mathbf{U}(\mathbf{1}_2)_3 \cong \mathbf{U}(1)$, which represents the translations, has the generator $iI(u, a)$

$$I(u, a) = I(u) - I(a^\star) = \int \frac{d^3q}{(2\pi)^3 q_0} \frac{[u(\vec{q})^\alpha, u(\vec{q})^\star_\alpha] - [a(\vec{q})^{\star\alpha}, a(\vec{q})_\alpha]}{2} = I(u, a)^\star$$

$$[I(u, a^\star) + \vec{S}(u, a^\star), I(u, a)] = 0 \tag{1.19}$$

The Fock inner product is positive with the stability group conjugation $\star$,

$$\langle [u(\vec{p})^\star_\alpha, u(\vec{q})^\beta] \rangle = \langle u(\vec{p})^\star_\alpha u(\vec{q})^\beta \rangle = \delta^\beta_\alpha (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \tag{1.20}$$

$$\langle [a(\vec{p})^{\star\beta}, a(\vec{q})_\alpha] \rangle = \langle a(\vec{p})^{\star\beta} a(\vec{q})_\alpha \rangle = \delta^\beta_\alpha (2\pi)^3 q_0 \delta(\vec{q} - \vec{p})$$

Quantization and Fock state lead to the familiar field anticommutators and Fock values of the commutators,

$$\begin{pmatrix} \{\mathbf{l}(0)^\star, \mathbf{l}(x)\} \\ \langle [\mathbf{l}(0)^\star, \mathbf{l}(x)] \rangle \end{pmatrix} = \rho_k \partial^k \begin{pmatrix} \mathbf{s}(x \mid m) \\ -i\mathbf{C}(x \mid m) \end{pmatrix}, \quad \{\mathbf{l}(0)^\star, \mathbf{l}(\vec{x})\} = \rho_0 \delta(\vec{x}), \quad \text{etc.} \tag{1.21}$$

Spinor fields are symmetric $\mathbf{l}^\dagger = \mathbf{l}$, $(i\mathbf{r})^\dagger = i\mathbf{r}$, etc., with respect to the indefinite conjugation exchanging particle creation with antiparticle annihilation

$$\text{conjugation } \dagger: \quad u(\vec{q})^\alpha \leftrightarrow a(\vec{q})^{\star\alpha}, \quad a(\vec{q})_\alpha \leftrightarrow u(\vec{q})^\star_\alpha \tag{1.22}$$

1.3. Weyl Particles

Weyl particles will be defined as massless Fermi particles $m = 0$ with stability group $\mathbf{U}(1)$ which may be called an internal (hyper)charge or circularity.

The massless limit of the $\mathbf{SL}(\mathbf{C}^2)_\mathbf{R}/\mathbf{SU}(2)$-transmutator, used for a Dirac field, leads to the two projectors for lightlike energy-momenta $q^2 = 0$, $q_0 \neq 0$:

$$p_+(q) = \lim_{m\to 0} \sqrt{\frac{m}{2q_0}}\, \lambda\left(\frac{q}{m}\right) = \frac{\mathbf{1}_2 + \vec{\sigma}\vec{q}/|\vec{q}\,|}{2}$$

$$p_-(q) = \lim_{m\to 0} \sqrt{\frac{m}{2q_0}}\, \hat{\lambda}\left(\frac{q}{m}\right) = \frac{\mathbf{1}_2 - \vec{\sigma}\vec{q}/|\vec{q}\,|}{2} \tag{1.23}$$

$$p_+(q_0, 0, 0, \pm q_0) = \frac{\mathbf{1}_2 \pm \sigma_3}{2} = p_-(q_0, 0, 0, \mp q_0)$$

Any spacelike direction $\vec{\sigma}\vec{q}/|\vec{q}\,|$ can be transformed into a fixed third axis σ_3 of a rest frame, determined up to $\mathbf{SO}(2)$ rotations of the (1, 2)-plane,

$$o\left(\frac{\vec{q}}{q_0}\right)\sigma_3 o\left(\frac{\vec{q}}{q_0}\right)^\star = \frac{\vec{\sigma}\vec{q}}{|\vec{q}\,|}, \qquad q_0^2 = \vec{q}^{\,2} > 0 \tag{1.24}$$

with a 'rotation' $o(\vec{q}/q_0) \in \mathbf{SU}(2)$ as a representative of a class in $\mathbf{SO}(3)/\mathbf{SO}(2) \cong \mathbf{SU}(2)/\mathbf{U}(1)_3$,

$$o\left(\frac{\vec{q}}{q_0}\right) = \frac{1}{\sqrt{2q_0(q_0 + q_3)}}\begin{pmatrix} q_0 + q_3 & q_1 - iq_2 \\ -q_1 - iq_2 & q_0 + q_3 \end{pmatrix}, \qquad o(0, 0, 1) = \mathbf{1}_2$$

$$p_\pm(q) = o\left(\frac{\vec{q}}{q_0}\right)\frac{\mathbf{1}_2 \pm \sigma_3}{2}\, o\left(\frac{\vec{q}}{q_0}\right)^\star = o_\pm\left(\frac{\vec{q}}{q_0}\right) o_\pm\left(\frac{\vec{q}}{q_0}\right)^\star \tag{1.25}$$

$$\text{with} \quad o_\pm\left(\frac{\vec{q}}{q_0}\right)^A = o\left(\frac{\vec{q}}{q_0}\right)_{1,2}^A$$

Therewith the time-space translation analysis of a free massless Weyl field with a left-handed Lorentz group representation and classical Lagrangian $\mathcal{L}(\mathbf{l}_+) = i\mathbf{l}_+\check{\rho}_k\partial^k\mathbf{l}_+^\star$, e.g., of the electron neutrino field in the standard model—if massless—looks as follows:

$$\mathbf{l}_+(x)^A = \int \frac{d^3q}{(2\pi)^3 q_0}\sqrt{q_0}\, o_+\left(\frac{\vec{q}}{q_0}\right)^A (e^{ixq}\mathrm{u}(\vec{q}) + e^{-ixq}\mathrm{a}(\vec{q})^\star) \tag{1.26}$$

$$\mathbf{l}_+(x)_{\dot{A}}^\star = \int \frac{d^3q}{(2\pi)^3 q_0}\sqrt{q_0}\, o_+\left(\frac{\vec{q}}{q_0}\right)_{\dot{A}}^\star (e^{ixq}\mathrm{a}(\vec{q}) + e^{-ixq}\mathrm{u}(\vec{q})^\star)$$

With the massless field stability group $\mathbf{U}(1_2) \times \mathbf{U}(1)_3 \subset \mathbf{U}(2)$ there is no $\mathbf{SU}(2)$-spin degree of freedom left for the particles.

The conjugation $\star$ exchanges creation with annihilation

$$\left.\begin{array}{r}\text{conjugation } \star\\ \text{for stability group } \mathbf{U}(1_2)\end{array}\right\} \quad \mathrm{u}(\vec{q}) \leftrightarrow \mathrm{u}(\vec{q})^\star, \quad \mathrm{a}(\vec{q}) \leftrightarrow \mathrm{a}(\vec{q})^\star \tag{1.27}$$

The stability group $\mathbf{U}(1_2) \cong \mathbf{U}(1)$ is generated by $iI_+(\mathrm{u}, \mathrm{a}^\star)$

$$\begin{aligned} I_+(\mathrm{u}, \mathrm{a}^\star) &= I_+(\mathrm{u}) + I_+(\mathrm{a}^\star)\\ &= \int \frac{d^3q}{(2\pi)^3 q_0} \frac{[\mathrm{u}(\vec{q}), \mathrm{u}(\vec{q})^\star] + [\mathrm{a}(\vec{q})^\star, \mathrm{a}(\vec{q})]}{2}\\ &= I_+(\mathrm{u}, \mathrm{a}^\star)^\star \end{aligned} \tag{1.28}$$

$I_+(\mathrm{u}, \mathrm{a}^\star)$ is an internal (hyper)charge or the polarization.

The translation representing group $\mathbf{U}(1)_3 \cong \mathbf{U}(1)$ is generated by $iI_+(\mathrm{u}, \mathrm{a})$,

$$\begin{aligned} I_+(\mathrm{u}, \mathrm{a}) &= I_+(\mathrm{u}) - I_+(\mathrm{a}^\star)\\ &= \int \frac{d^3q}{(2\pi)^3 q_0} \frac{[\mathrm{u}(\vec{q}), \mathrm{u}(\vec{q})^\star] - [\mathrm{a}(\vec{q})^\star, \mathrm{a}(\vec{q})]}{2} = I_+(\mathrm{u}, \mathrm{a})^\star \end{aligned} \tag{1.29}$$

$$[I_+(\mathrm{u}, \mathrm{a}^\star), I_+(\mathrm{u}, \mathrm{a})] = 0$$

The Fock product is positive with the conjugation $\star$, in analogy to the massive case.

1.4. Majorana Particles

Majorana particles, if they exist, will be defined as Fermi particles with nontrivial mass $m > 0$ and stability group $\mathbf{SU}(2)$ for spin $s = 1/2$ without an internal charge.

Since the $\mathbf{SL}(\mathbb{C}^2)_\mathbb{R}$ Lorentz properties of the irreducible Weyl contributions $\mathbf{l}(x)^A$ and $\mathbf{r}(x)^\star_{\dot{A}}$ in a Dirac field are isomorphic with the invariant bilinear spinor 'metric'

$$\lambda \in \mathbf{SL}(\mathbb{C}^2)_\mathbb{R}: \quad \epsilon_{AB}\lambda^B_C\epsilon^{CD} = (\lambda^{-1})^D_A, \qquad \epsilon_{AB} = -\epsilon_{BA} \tag{1.30}$$

one can consider the case where the four Dirac fields $(\mathbf{l}, \mathbf{r}^\star; \mathbf{r}, \mathbf{l}^\star)$ are built with only two irreducible left- and right-handed Weyl representations $(\mathbf{L}, \mathbf{R})$ by 'crossover' identifying particles and antiparticles

$$\mathrm{a}(\vec{q})^{\star\alpha} = i\epsilon^{\alpha\beta}\mathrm{u}(\vec{q})^\star_\beta, \qquad \mathrm{a}(\vec{q})_\alpha = -i\mathrm{u}(\vec{q})^\beta\epsilon_{\beta\alpha} \tag{1.31}$$

Therewith one describes Majorana fields with the time-space translation analysis

$$\mathbf{L}(x)^A = \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{m}\,\lambda\left(\frac{q}{m}\right)^A_\alpha \frac{e^{ixq}\mathrm{u}(\vec{q})^\alpha + e^{-ixq} i\epsilon^{\alpha\beta}\mathrm{u}(\vec{q})^\star_\beta}{\sqrt{2}} = i\epsilon^{AB}\mathbf{R}(x)^\star_{\dot{B}}$$

$$\mathbf{L}(x)^\star_{\dot{A}} = \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{m}\,\lambda\left(\frac{q}{m}\right)^{\star\alpha}_{\dot{A}} \frac{-e^{ixq}\mathrm{u}(\vec{q})^\beta i\epsilon_{\beta\alpha} + e^{-ixq}\mathrm{u}(\vec{q})^\star_\alpha}{\sqrt{2}} = -i\mathbf{R}(x)^{\dot{B}}\epsilon_{\dot{B}\dot{A}} \tag{1.32}$$

The classical Lagrangian is only $\mathbf{SL}(\mathbb{C}^2)_\mathbf{R}$-invariant,

$$\mathcal{L}(\mathbf{L}) = i\mathbf{L}\check{\rho}_k\partial^k\mathbf{L}^\star - \mathcal{I}(\mathbf{L}) \tag{1.33}$$

$$\mathcal{I}(\mathbf{L}) = im(\epsilon_{BA}\mathbf{L}^A\mathbf{L}^B - \mathbf{L}^\star_{\dot{A}}\mathbf{L}^\star_{\dot{B}}\epsilon^{\dot{B}\dot{A}})$$

The two conjugations use the two components $\alpha = 1, 2$,

$$\left.\begin{array}{r}\text{conjugation } \star \\ \text{for stability group } \mathbf{SU}(2)\end{array}\right\} \quad \mathrm{u}(\vec{q})^\alpha \leftrightarrow \delta^{\alpha\beta}\mathrm{u}(\vec{q})^\star_\beta \tag{1.34}$$

$$\text{conjugation } \dagger \quad \mathrm{u}(\vec{q})^\alpha \leftrightarrow i\epsilon^{\alpha\beta}\mathrm{u}(\vec{q})^\star_\beta$$

One can write for the combinations in the time-space translation analysis

$$\mathrm{u}^1 = \mathrm{u}, \quad \mathrm{u}^2 = i\mathrm{a} \Rightarrow \begin{cases} \mathrm{u}^\alpha + i\epsilon^{\alpha\beta}\mathrm{u}^\star_\beta \cong \begin{pmatrix} \mathrm{u} + \mathrm{a}^\star \\ i(\mathrm{a} - \mathrm{u}^\star) \end{pmatrix} \\ i(\mathrm{u}^\alpha - i\epsilon^{\alpha\beta}\mathrm{u}^\star_\beta) \cong \begin{pmatrix} i(\mathrm{u} - \mathrm{a}^\star) \\ -(\mathrm{a} + \mathrm{u}^\star) \end{pmatrix} \end{cases} \tag{1.35}$$

The dual pair quantization and Fock values are analogous to the Dirac case

$$\{\mathrm{u}(\vec{p})^\star_\alpha, \mathrm{u}(\vec{q})^\beta\} = \delta^\beta_\alpha(2\pi)^3 q_0\delta(\vec{q} - \vec{p}) = \langle[\mathrm{u}(\vec{p})^\star_\alpha, \mathrm{u}(\vec{q})^\beta]\rangle = \langle \mathrm{u}(\vec{p})^\star_\alpha \mathrm{u}(\vec{q})^\beta\rangle \tag{1.36}$$

The generators $i\vec{S}(\mathrm{u})$ for the spin group $\mathbf{SU}(2)$ and $iI(\mathrm{u})$ for the translations realizing group $\mathbf{U}(1)$ are

$$\vec{S}(\mathrm{u}) = \int \frac{d^3q}{(2\pi)^3 q_0} \vec{\sigma}^\beta_\alpha \frac{[\mathrm{u}(\vec{q})^\alpha, \mathrm{u}(\vec{q})^\star_\beta]}{2} = \vec{S}(\mathrm{u})^\star$$

$$I(\mathrm{u}) = \int \frac{d^3q}{(2\pi)^3 q_0} \frac{[\mathrm{u}(\vec{q})^\alpha, \mathrm{u}(\vec{q})^\star_\alpha]}{2} = I(\mathrm{u})^\star \tag{1.37}$$

$$[\vec{S}(\mathrm{u}), I(\mathrm{u})] = 0$$

2. FIELDS WITH INDEFINITE METRIC

In this section the less familiar indefinite metric quantum fields are discussed. Such fields cannot be expanded with particles only, i.e., with translation eigenvectors. They are used for locally formulated relativistic invariant interactions.

Prominent examples for indefinite metric quantum fields are gauge fields: The electromagnetic vector field with its four Lorentz components has two particle degrees of freedom with positive-definite modality group $\mathbf{U}(2)$, the two massless photons as left- and right-polarized representations of the stability group $\mathbf{SO}(2)$ for the Witt decomposition $\mathrm{M} \cong \mathrm{T} \oplus \mathrm{S}^2 \oplus \mathrm{S}^1$ of the time-space translations. The two additional $\mathbf{SO}(2)$-trivial lightlike degrees of freedom $\mathrm{T} \oplus \mathrm{S}^1 \cong \mathrm{L}_+ \oplus \mathrm{L}_- \cong \mathbb{R}^2$ have no particle interpretation. They describe the gauge degree of freedom and the Coulomb interaction and have an indefinite $\mathbf{U}(1, 1)$-modality group wherein the translations are represented.

Also, Fadeev–Popov fields have an indefinite $\mathbf{U}(1, 1)$-conjugation $\times$ without particle interpretation.

Characteristic structures of indefinite metric quantum fields are derived Dirac measures. A relativistic field $\mathbf{\Phi}'(x|m) = (d/dm^2)\mathbf{\Phi}(x|m)$ which is conjugation $*$ symmetric and allows an analysis of the time-space translations with a derived Dirac measure for mass $m \geq 0$

$$\mathbf{\Phi}'_{\pm}(x|m) = \int \frac{d^4q\, e^{ixq}}{(2\pi)^3} \begin{pmatrix} 1 \\ -i\epsilon(q_0) \end{pmatrix} \delta'(m^2 - q^2)\mathrm{e}(q) = \mathbf{\Phi}'_{\pm}(x|m)^* \quad (2.1)$$

contains harmonic components $\mathrm{e}(q, x)$ with a first-order polynomial dependence in the time-space translations

$$\mathbf{\Phi}'_{\pm}(x|m) = \int \frac{d^3q}{(2\pi)^3 q_0} \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{e^{ixq}\mathrm{e}(q, x) \pm e^{-ixq}\mathrm{e}(q, x)^*}{2}\Bigg|_{q_0=(m^2+\vec{q}^2)^{1/2}} \quad (2.2)$$

$$\text{with} \quad \mathrm{e}(q, x) = \mathrm{e}_0(q) + ix\mathrm{e}_1(q) = \mathrm{e}(-x, -q)^*$$

In this case, the real 4-dimensional additive group of the time-space translations $\mathbf{M} \cong \mathbb{R}^4$ is represented as a subgroup of the indefinite unitary group $\mathbf{U}(2, 2)$ with the energy-momenta q, $q^2 = m^2$, as eigenvalues

$$D_2(\cdot|q): \quad \mathrm{M} \to \mathbf{U}(2, 2), \quad \begin{cases} D_2(x|q) = e^{ixq}\begin{pmatrix} \mathbf{1}_2 & i\rho^j x_j \\ 0 & \mathbf{1}_2 \end{pmatrix} = e^{iQ^j x_j} = D_2(-x|q)^\times \\ \partial^j|_{x=0} D_2(x|q) = iQ^j = i\begin{pmatrix} q^j\mathbf{1}_2 & \rho^j \\ 0 & q^j\mathbf{1}_2 \end{pmatrix} \\ \rho^j = (\mathbf{1}_2, \vec{\sigma}) \end{cases} \quad (2.3)$$

The image of the time-space translations is an $\mathbf{R}^4$-isomorphic unitary subgroup of $\mathbf{U}(2, 2)$ as illustrated by the triangular Jordan matrix with the characteristic nilpotent contributions. The time-space representations $D_2(x \mid q)$ are faithful and reducible, but nondecomposable. Because of the indefinite unitary modality group $\mathbf{U}(2, 2)$ such fields have no probability interpretation in terms of particles.

2.1. Maxwell–Witt Fields

Maxwell–Witt fields (Saller *et al.*, 1995) will be defined as massless Lorentz vector Bose fields with stability group $\mathbf{SO}(2)$ for circularity (polarization). In addition to massless particles they contain also nonparticle contributions.

The $\mathbf{SO}^+(1, 3)$-invariant Lagrangian for a free massless vector field, e.g., the electromagnetic field

$$\mathscr{L}(\mathbf{A}, \mathbf{F}, \mathbf{G}) = \mathbf{G}\partial_k\mathbf{A}^k + \mathbf{F}^{jk}\frac{\partial_j\mathbf{A}_k - \partial_k\mathbf{A}_j}{2} - \mathscr{H}(\mathbf{A}, \mathbf{F}, \mathbf{G}) \qquad (2.4)$$

$$\mathscr{H}(\mathbf{A}, \mathbf{F}, \mathbf{G}) = -\mu\frac{\mathbf{F}^{jk}\mathbf{F}_{jk}}{4} - \sigma\frac{\mathbf{G}^2}{2}$$

has to include, with respect to a quantum framework, a canonical partner $\mathbf{G}$, called a gauge-fixing field, for the scalar part of the vector field $\mathbf{A}^k$. Here $\mu > 0$ is a mass (no particle mass) which, in an interacting theory, can be related to the gauge coupling constant; $\sigma \neq 0$ is called the gauge-fixing constant.

In the quantization distributions (Nakanishi and Ojima, 1990)

$$\begin{pmatrix} [i\mathbf{F}^{kj}(0), \mathbf{A}_r(x)] \\ [\mathbf{A}^k(0), \mathbf{G}(x)] \\ [\mathbf{A}^k(0), \mathbf{A}^j(x)] \end{pmatrix} \qquad (2.5)$$

$$= \int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\epsilon(q_0)\begin{pmatrix} \epsilon^{kj}_{lr}q^l\delta(q^2) \\ q^k\delta(q^2) \\ -\mu\eta^{kj}\delta(q^2) - (\mu + \sigma)q^kq^j\delta'(q^2) \end{pmatrix}$$

there arises the derived Dirac measure $\delta'(q^2)$ as a characteristic feature of the nonparticle structure. Its time translation analysis looks as follows:

$$\mathbf{s}'(x \mid m) = \frac{d}{dm^2}\mathbf{s}(x \mid m) = -i\int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\epsilon(q_0)\delta'(m^2 - q^2) \qquad (2.6)$$

$$= \int \frac{d^3q\, e^{-i\vec{x}\vec{q}}}{(2\pi)^3 q_0}\frac{x_0q_0\cos x_0q_0 - \sin x_0q_0}{2q_0^2}$$

The translation analysis of the massless vector field has to include a transmutation $O(\vec{q}/q_0)$ with $q^2 = 0$, $q \neq 0$, from the rest frame stability group

SO(3) to SO(2) for rest frames with fixed third axis

$$O\left(\frac{\vec{q}}{q_0}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{(q_1)^2}{q_0(q_0 + q_3)} & -\frac{q_1 q_2}{q_0(q_0 + q_3)} & \frac{q_1}{q_0} \\ 0 & -\frac{q_1 q_2}{q_0(q_0 + q_3)} & 1 - \frac{(q_2)^2}{q_0(q_0 + q_3)} & \frac{q_2}{q_0} \\ 0 & -\frac{q_1}{q_0} & -\frac{q_2}{q_0} & \frac{q_3}{q_0} \end{pmatrix} \tag{2.7}$$

$$O\left(\frac{\vec{q}}{q_0}\right)^k_j = \frac{1}{2} \operatorname{tr} o\left(\frac{\vec{q}}{q_0}\right) \rho^k o\left(\frac{\vec{q}}{q_0}\right)^\star \check{\rho}_j, \qquad O(0, 0, 1) = \mathbf{1}_4$$

According to the isomorphy of a time-space and light-light decomposition $\mathbb{T} \oplus \mathbb{S}^1 \cong \mathbb{L}_+ \oplus \mathbb{L}_-$, it is convenient to transform from a time-space Sylvester basis with diagonal metrical tensor η to a light-space-light Witt basis with 'skew-diagonal' metrical tensor ι

$$\text{Sylvester:} \quad -\eta = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix}, \qquad \text{Witt:} \quad -\iota = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{1}_2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.8}$$

$$\iota = w\, \eta\, w^T \qquad \text{with} \quad w = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & \mathbf{1}_2 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

For vector fields, the Lorentz group with its signature (1, 3) indefinite 'metric' is represented in the indefinite unitary group $\mathbf{SO}^+(1, 3) \subset \mathbf{U}(1, 3)$ which determines the conjugations and modality groups $\mathbf{U}(1, 1) \circ \mathbf{U}(2) \subset \mathbf{U}(1, 3)$ for the gauge field. A massless vector field has the time-space translation analysis (Saller *et al.*, 1995)

$$\mathbf{A}(x)^k = \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{\mu}\, O\left(\frac{\vec{q}}{q_0}\right)^k_j w^j_{\cdots} \begin{pmatrix} \frac{e^{ixq}\mathbf{B}(\vec{q}, x_0) + e^{-ixq} N_0 \mathbf{G}(\vec{q})^\times}{\sqrt{2}} \\ \frac{e^{ixq}\mathbf{U}(\vec{q})^1 + e^{-ixq}\mathbf{U}(\vec{q})^\star_1}{\sqrt{2}} \\ \frac{e^{ixq}\mathbf{U}(\vec{q})^2 + e^{-ixq}\mathbf{U}(\vec{q})^\star_2}{\sqrt{2}} \\ \frac{e^{ixq} N_0 \mathbf{G}(\vec{q}) + e^{-ixq}\mathbf{B}(\vec{q}, x_0)^\times}{\sqrt{2}} \end{pmatrix} \tag{2.9}$$

$$\mathbf{G}(x) = i \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{\mu}\, \frac{e^{ixq}\mathbf{G}(\vec{q}) - e^{-ixq}\mathbf{G}(\vec{q})^\times}{\sqrt{2}}$$

The (1, 2)-components $U(\vec{q})^{1,2}$ have time translation representations in the positive-definite group **U**(2); they describe particle degrees of freedom (photons). The (0, 3)-components $(B(\vec{q}), G(\vec{q}))$ have a linear translation dependence

$$B(\vec{q}, x_0) = B(\vec{q}) + \frac{ix_0q_0}{M_0} G(\vec{q}) \quad \text{with} \quad \begin{cases} \dfrac{1}{M_0} = -\dfrac{\mu + \sigma}{\mu} \\ N_0 = \dfrac{3\mu + \sigma}{\mu} \end{cases} \tag{2.10}$$

The characteristic terms $(ix_0q_0/M_0)e^{ix_0q_0}$ are associated with nondecomposable, but reducible representations (Boerner, 1955; Saller, 1989) of the time translations

$$D_2(x_0 | q_0) = [\exp(ix_0q_0)]\begin{pmatrix} 1 & ix_0q_0/M_0 \\ 0 & 1 \end{pmatrix} = \exp\left[ix_0q_0\begin{pmatrix} 1 & 1/M_0 \\ 0 & 1 \end{pmatrix}\right] \tag{2.11}$$

as an R-isomorphic subgroup of the indefinite unitary group **U**(1, 1)

$$D_2(x_0 | q_1)^{\times} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_2(x_0 | q_0)^{\star} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D_2(-x_0 | q_0) \tag{2.12}$$

The quantization connects dual pairs

$$\text{for } (1, 2)\text{:} \quad [U(\vec{p})^{\star}_{\alpha}, U(\vec{q})^{\beta}] = \delta^{\beta}_{\alpha}(2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \tag{2.13}$$

$$\text{for } (0, 3)\text{:} \quad \begin{cases} [G(\vec{p})^{\times}, B(\vec{q})] = [B(\vec{p})^{\times}, G(\vec{q})] = (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \\ [G(\vec{p})^{\times}, G(\vec{q})] = 0 = [B(\vec{p})^{\times}, B(\vec{q})] \end{cases}$$

The (1, 2)-particle degrees of freedom have the positive **U**(1)-conjugation $\star$, whereas the indefinite **U**(1, 1)-conjugation $\times$ applies for the (0, 3)-nonparticle degrees of freedom

$$\left.\begin{array}{c} \text{conjugation } \star \\ \text{for (1, 2)-modality group } \mathbf{U}(1_2) \end{array}\right\} U(\vec{q})^{1,2} \leftrightarrow U(\vec{q})^{\star}_{1,2} \tag{2.14}$$

$$\left.\begin{array}{c} \text{conjugation } \times \\ \text{for (0, 3)-modality group } \mathbf{U}(1, 1) \end{array}\right\} \begin{cases} G(\vec{q}) \leftrightarrow G(\vec{q})^{\times} \\ B(\vec{q}) \leftrightarrow B(\vec{q})^{\times} \end{cases}$$

The time translations group, as R-isomorphic noncompact subgroup of $\mathbf{U}(1_2) \circ \mathbf{U}(1, 1) \subset \mathbf{U}(1, 3)$, is generated with

$$
\begin{aligned}
H(\mathrm{U}, \mathrm{B}, \mathrm{G}) &= \int \frac{d^3q}{(2\pi)^3 q_0} \left(\frac{\{\mathrm{U}(\vec{q})^\alpha, \mathrm{U}(\vec{q})^\star_\alpha\} + \{\mathrm{B}(\vec{q}), \mathrm{G}(\vec{q})^\times\} + \{\mathrm{G}(\vec{q}), \mathrm{B}(\vec{q})^\times\}}{2} \right. \\
&\quad \left. + \frac{\mathrm{G}(\vec{q})\mathrm{G}(\vec{q})^\times}{M_0} \right) \qquad (2.15) \\
&= I(\mathrm{U}) + H(\mathrm{B}, \mathrm{G}) = I(\mathrm{U})^\star + H(\mathrm{B}, \mathrm{G})^\times
\end{aligned}
$$

The stability group $\mathbf{SO}(2) \cong \mathbf{U}(1) \subset \mathbf{SU}(2)$ (polarization) is generated by $iS(\mathrm{U})$ with the particle degrees of freedom only

$$
S(\mathrm{U}) = \int \frac{d^3q}{(2\pi)^3 q_0} \frac{\{\mathrm{U}(\vec{q})^1, \mathrm{U}(\vec{q})^\star_1\} - \{\mathrm{U}(\vec{q})^2, \mathrm{U}(\vec{q})^\star_2\}}{2} = S(\mathrm{U})^\star
$$

$$
[H(\mathrm{U}, \mathrm{B}, \mathrm{G}), S(\mathrm{U})] = 0 \qquad (2.16)
$$

With the U(1)-conjugation $\star$, the Fock product for the particle degrees of freedom is positive definite

$$
\text{for } (1, 2): \quad \langle \{\mathrm{U}(\vec{p})^\star_\alpha, \mathrm{U}(\vec{q})^\beta\} \rangle = \delta^\beta_\alpha (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) = \langle \mathrm{U}(\vec{p})^\star_\alpha \mathrm{U}(\vec{q})^\beta \rangle \qquad (2.17)
$$

The U(1, 1)-conjugation $\times \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for the nonparticle degrees of freedom leads to an indefinite inner Fock product

$$
\text{for } (0, 3): \quad \begin{cases} \langle \{\mathrm{G}(\vec{p})^\times, \mathrm{B}(\vec{q})\} \rangle = \langle \{\mathrm{B}(\vec{p})^\times, \mathrm{G}(\vec{q})\} \rangle = (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \\ \Rightarrow \left\langle \dfrac{\mathrm{G}(\vec{p})^\times \pm \mathrm{B}(\vec{p})^\times}{\sqrt{2}} \dfrac{\mathrm{G}(\vec{q}) \pm \mathrm{B}(\vec{q})}{\sqrt{2}} \right\rangle = \pm (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \end{cases} \qquad (2.18)
$$

For a probability interpretation, the indefinite metric has to be avoided for the asymptotic state space: Fadeev–Popov fields counterbalance the 'negative probabilities.' The requirement of gauge invariance like Becchi–Rouet–Stora invariance in a quantum theory projects to a positive-definite asymptotic particle subspace (Section 4).

2.2. Fadeev–Popov Fields

Fadeev–Popov fields will be defined as massless Lorentz scalar Fermi fields. They have no particle contributions.

Their classical Lagrangian uses two scalar fields $\mathbf{A}_+$, $\mathbf{U}_-$ in a second-order derivative formalism $\mathcal{L}(\mathbf{A}_+, \mathbf{U}_-) = i(\partial^k \mathbf{A}_+)(\partial_k \mathbf{U}_-)$ or, in addition, two vector fields $\mathbf{U}_+^k$, $\mathbf{A}_-^k$ for a first-order formulation,

$$\begin{aligned}\mathcal{L}(\mathbf{A}_\pm, \mathbf{U}_\pm) &= i\mathbf{A}_+\partial_k \mathbf{U}_+^k + i\mathbf{U}_-\partial_k \mathbf{A}_-^k - \mathcal{H}(\mathbf{A}_\pm, \mathbf{U}_\pm) \\ \mathcal{H}(\mathbf{A}_\pm, \mathbf{U}_\pm) &= i\mu \mathbf{U}_+^k \mathbf{A}_{-k}\end{aligned} \tag{2.19}$$

with a mass unit $\mu > 0$ (no particle mass).

The quantization for the Fadeev–Popov fields with the translation analysis

$$\begin{aligned}\mathbf{A}_+(x) &= \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{\mu}\, \frac{e^{ixq}\mathrm{a}(\vec{q}) + e^{-ixq}\mathrm{a}(\vec{q})^\times}{\sqrt{2}} \\ \mathbf{U}_-(x) &= i\int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{\mu}\, \frac{e^{ixq}\mathrm{u}(\vec{q}) - e^{-ixq}\mathrm{u}(\vec{q})^\times}{\sqrt{2}} \\ \mathbf{U}_+(x)^k &= \int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{\mu}\, \Lambda\left(\frac{q}{\mu}\right)_0^k \frac{e^{ixq}\mathrm{u}(\vec{q}) + e^{-ixq}\mathrm{u}(\vec{q})^\times}{\sqrt{2}} \\ \mathbf{A}_-(x)^k &= i\int \frac{d^3q}{(2\pi)^3 q_0} \sqrt{\mu}\, \Lambda\left(\frac{q}{\mu}\right)_0^k \frac{e^{ixq}\mathrm{a}(\vec{q}) - e^{-ixq}\mathrm{a}(\vec{q})^\times}{\sqrt{2}}\end{aligned} \tag{2.20}$$

connects as dual pairs

$$\{\mathrm{u}(\vec{p})^\times, \mathrm{a}(\vec{q})\} = (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) = \{\mathrm{a}(\vec{p})^\times, \mathrm{u}(\vec{q})\} \tag{2.21}$$

A positive U(1)-conjugation $\star$ is impossible, i.e., $\mathrm{u}^\times$ and $\mathrm{a}^\times$ cannot be identified with $\mathrm{a}^\star$ and $\mathrm{u}^\star$, resp. With $\{\mathbf{U}_-, \mathbf{U}_-\} = 0$ also an identification u = a and $\mathrm{u}^\times = \mathrm{u}^\star$ cannot be used.

Therewith Faddeev–Popov fields have an indefinite conjugation

$$\left.\begin{matrix}\text{conjugation } \times \\ \text{for modality group } \mathbf{U}(1, 1)\end{matrix}\right\} \quad \mathrm{u}(\vec{q}) \leftrightarrow \mathrm{u}(\vec{q})^\times, \quad \mathrm{a}(\vec{q}) \leftrightarrow \mathrm{a}(\vec{q})^\times \tag{2.22}$$

The fields are symmetric with the conjugation $\times$, i.e., $\mathbf{U}_- = \mathbf{U}_-^\times$, etc. (Rudolph and Dürr, 1972; Kugo and Ojima, 1978).

The time translations are generated by $iI(\mathrm{a}, \mathrm{u})$ with

$$I(\mathrm{a}, \mathrm{u}) = \int \frac{d^3q}{(2\pi)^3 q_0} \frac{[\mathrm{a}(\vec{q}), \mathrm{u}(\vec{q})^\times] + [\mathrm{u}(\vec{q}), \mathrm{a}(\vec{q})^\times]}{2} = I(\mathrm{a}, \mathrm{u})^\times \tag{2.23}$$

The Fock inner product is indefinite with the U(1, 1)-conjugation $\times \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\langle[\mathrm{u}(\vec{p})^\times, \mathrm{a}(\vec{q})]\rangle = \langle[\mathrm{a}(\vec{p})^\times, \mathrm{u}(\vec{q})]\rangle = (2\pi)^3 q_0 \delta(\vec{q} - \vec{p})$$
$$\Rightarrow \left\langle \frac{\mathrm{u}(\vec{p})^\times \pm \mathrm{a}(\vec{p})^\times}{\sqrt{2}} \frac{\mathrm{u}(\vec{q}) \pm \mathrm{a}(\vec{q})}{\sqrt{2}} \right\rangle = \pm(2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \quad (2.24)$$

2.3. Heisenberg–Majorana Fields

It is possible, in analogy to the Majorana, Dirac, and Weyl particle fields of Section 1 with U(1) translation representation, to construct Heisenberg–Majorana, Heisenberg–Dirac, and Heisenberg–Weyl fields, all with the translations realized in the indefinite unitary modality group U(2, 2). Those fields have no particle interpretation, but may be used for the implementation of interactions.

Left-handed Heisenberg–Majorana fields $\mathbf{b}^A$, $\mathbf{g}^A$ are analyzable with the time-space translations represented in U(2, 2),

$$\begin{aligned}
\mathbf{b}(x)^A &= \int \frac{d^3q}{(2\pi)^3 q_0} \lambda\left(\frac{q}{m}\right)^A_\alpha \frac{e^{ixq}\mathrm{b}(\vec{q}, x)^\alpha + e^{-ixq} i\epsilon^{\alpha\beta}\mathrm{b}(\vec{q}, x)^\times_\beta}{\sqrt{2}} \\
\mathbf{b}(x)^\times_{\dot{A}} &= \int \frac{d^3q}{(2\pi)^3 q_0} \lambda\left(\frac{q}{m}\right)^{\star\alpha}_{\dot{A}} \frac{-e^{ixq}\mathrm{b}(\vec{q}, x)^\beta i\epsilon_{\beta\alpha} + e^{-ixq}\mathrm{b}(\vec{q}, x)^\times_\alpha}{\sqrt{2}} \\
\mathbf{g}(x)^A &= \int \frac{d^3q}{(2\pi)^3 q_0} \lambda\left(\frac{q}{m}\right)^A_\alpha \frac{e^{ixq}\mathrm{g}(\vec{q})^\alpha + e^{-ixq} i\epsilon^{\alpha\beta}\mathrm{g}(\vec{q})^\times_\beta}{\sqrt{2}} \\
\mathbf{g}(x)^\times_{\dot{A}} &= \int \frac{d^3q}{(2\pi)^3 q_0} \lambda\left(\frac{q}{m}\right)^{\star\alpha}_{\dot{A}} \frac{-e^{ixq}\mathrm{g}(\vec{q})^\beta i\epsilon_{\beta\alpha} + e^{-ixq}\mathrm{g}(\vec{q})^\times_\alpha}{\sqrt{2}}
\end{aligned} \quad (2.25)$$

The harmonic components have a linear time-space dependence with the translation components $x(q/m)_k$, $k = 0, 1, 2, 3$, written in a rest system

$$\begin{aligned}
\mathrm{b}(\vec{q}, x)^\alpha &= \mathrm{b}(\vec{q})^\alpha + ix\left(\frac{q}{m}\right)^\alpha_\beta \mathrm{g}(\vec{q})^\beta \\
x\left(\frac{q}{m}\right)^\alpha_\beta &= (\rho^k)^\alpha_\beta x\left(\frac{q}{m}\right)_k = \lambda\left(\frac{q}{m}\right)^{-1\alpha}_A x^A_{\dot{A}}\, \hat{\lambda}\left(\frac{q}{m}\right)^{\dot{A}}_\beta \\
x\left(\frac{q}{m}\right)_k &= \Lambda\left(\frac{q}{m}\right)^{-1j}_k x_j, \qquad x^A_{\dot{A}} = (\rho^k)^A_{\dot{A}} x_k
\end{aligned} \quad (2.26)$$

The quantization of the harmonic components connects dual pairs

$$\{b(\vec{p})^{\times}_{\alpha}, g(\vec{q})^{\beta}\} = \{g(\vec{p})^{\times}_{\alpha}, b(\vec{q})^{\beta}\} = \delta^{\beta}_{\alpha}(2\pi)^3 q_0 \delta(\vec{q} - \vec{p})$$
$$\{g(\vec{p})^{\times}_{\alpha}, g(\vec{q})^{\beta}\} = 0 = \{b(\vec{p})^{\times}_{\alpha}, b(\vec{q})^{\beta}\} \tag{2.27}$$

and leads to the field quantization

$$\{\mathbf{b}(0)^{\times}_{\dot{A}}, \mathbf{b}(x)^{A}\} = -(\rho^k)^{A}_{\dot{A}} x_k \mathbf{s}(x \mid m), \qquad \{\mathbf{g}(0)^{\times}_{\dot{A}}, \mathbf{g}(x)^{A}\} = 0 \tag{2.28}$$
$$\{\mathbf{g}(0)^{\times}_{\dot{A}}, \mathbf{b}(x)^{A}\} = \{\mathbf{b}(0)^{\times}_{\dot{A}}, \mathbf{g}(x)^{A}\} = (\rho^k)^{A}_{\dot{A}} \partial_k\, \mathbf{s}(x \mid m)$$

with the characteristic derived measure

$$\frac{x_k}{2}\,\mathbf{s}(x \mid m) = \partial_k \mathbf{s}'(x \mid m) = \frac{d}{dm^2}\,\partial_k \mathbf{s}(x \mid m)$$
$$= \int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\, \epsilon(q_0) q_k \delta'(m^2 - q^2) \tag{2.29}$$

A classical $\mathbf{SL}(\mathbf{C}^2)_{\mathbf{R}}$-invariant Lagrangian reads

$$\mathscr{L}(\mathbf{b}, \mathbf{g}) = i\mathbf{b}\check{\rho}_k \partial^k \mathbf{g}^{\times} + i\mathbf{g}\check{\rho}_k \partial^k \mathbf{b}^{\times} - \mathscr{H}(\mathbf{b}, \mathbf{g}) \tag{2.30}$$
$$\mathscr{H}(\mathbf{b}, \mathbf{g}) = i(\epsilon_{BA}\mathbf{g}^A\mathbf{g}^B - \mathbf{g}^{\times}_{\dot{A}}\mathbf{g}^{\times}_{\dot{B}}\epsilon^{\dot{B}\dot{A}}) + im(\epsilon_{BA}\mathbf{b}^A\mathbf{g}^B - \mathbf{g}^{\times}_{\dot{A}}\mathbf{b}^{\times}_{\dot{B}}\epsilon^{\dot{B}\dot{A}})$$

The conjugation $\times$ for the time-space translations is characterized by the indefinite unitary group $\mathbf{U}(2, 2)$

$$\left.\begin{matrix}\text{conjugation } \times \\ \text{for modality group} \\ \mathbf{U}(2,2)\end{matrix}\right\} \quad \begin{cases} b(\vec{q})^{\alpha} \leftrightarrow \delta^{\alpha\beta} b(\vec{q})^{\times}_{\beta} \\ g(\vec{q})^{\alpha} \leftrightarrow \delta^{\alpha\beta} g(\vec{q})^{\times}_{\beta} \end{cases} \tag{2.31}$$

The $\mathbf{R}^4$-isomorphic time-space translation group is generated by $iQ(\mathrm{b}, \mathrm{g})^j$,

$$Q(\mathrm{b}, \mathrm{g})^j = \int \frac{d^3q}{(2\pi)^3 q_0} \left(q^j\, \frac{[b(\vec{q})^{\alpha}, g(\vec{q})^{\times}_{\alpha}] + [g(\vec{q})^{\alpha}, b(\vec{q})^{\times}_{\alpha}]}{2} + g(\vec{q})^{\alpha}(\rho^j)^{\beta}_{\alpha} g(\vec{q})^{\times}_{\beta} \right)$$
$$= I(\mathrm{b}, \mathrm{g})^j + N(\mathrm{g})^j = Q(\mathrm{b}, \mathrm{g})^{j\times} \tag{2.32}$$

A compatible stability group $\mathbf{U}(1)$ is generated by $iI(\mathrm{b}, \mathrm{g})$

$$I(\mathrm{b}, \mathrm{g}) = \int \frac{d^3q}{(2\pi)^3 q_0}\, \frac{[b(\vec{q})^{\alpha}, g(\vec{q})^{\times}_{\alpha}] + [g(\vec{q})^{\alpha}, b(\vec{q})^{\times}_{\alpha}]}{2} = I(\mathrm{b}, \mathrm{g})^{\times} \tag{2.33}$$
$$[Q(\mathrm{b}, \mathrm{g})^j, I(\mathrm{b}, \mathrm{g})] = 0$$

The fields are symmetric under the conjugation $\dagger$, i.e., $\mathbf{b}^{\dagger} = \mathbf{b}$, etc.,

$$\text{conjugation } \dagger \quad \begin{cases} b(\vec{q})^{\alpha} \leftrightarrow i\epsilon^{\alpha\beta} b(\vec{q})^{\times}_{\beta} \\ g(\vec{q})^{\alpha} \leftrightarrow i\epsilon^{\alpha\beta} g(\vec{q})^{\times}_{\beta} \end{cases} \tag{2.34}$$

3. MODALITY GROUPS—THE MATHEMATICS

In this section the positive and indefinite unitary representations of the time-space translations are discussed together with their implementation in quantum algebras. The mathematical structures of this section have been used implicitly in the former two sections. They are exhibited rather sketchily in the following—more as a glossary—and can be found in more detail in the literature (Bourbaki, 1959, Boerner, 1955; Saller, 1989, 1993a,c).

3.1. Conjugations and Unitary Groups

A conjugation * is an antilinear isomorphism between a complex vector space $V \cong \mathbb{C}^d$ and the vector space $V^T \cong \mathbb{C}^d$ of its linear forms. It defines a nondegenerate sesquilinear form which, for a conjugation, is required to be symmetric

$$\begin{aligned} &\text{conjugation:} && *:\quad V \leftrightarrow V^T, \quad v, \omega^* \leftrightarrow v^*, \omega \\ &\text{dual product:} && V^T \times V \to \mathbb{C}, \quad (\omega, u) \mapsto \omega(u) = \langle \omega, u\rangle \\ &\text{inner product:} && *\langle\cdot|\cdot\rangle:\quad V \times V \to \mathbb{C}, \quad *\langle v|u\rangle = \langle v^*, u\rangle = \overline{\langle u^*, v\rangle} \end{aligned} \tag{3.1}$$

In the opposite direction, each symmetric nondegenerate sesquilinear form of a complex vector space $V \cong \mathbb{C}^d$ determines a conjugation.

With the conjugation defined between the vector space and its dual, a conjugation is defined on all multilinear structures, e.g., on the V-endomorphisms $V \oplus V^T$ by $(v\omega)^* = \omega^* v^*$, etc.

Since any conjugation * on $V \cong \mathbb{C}^d$ determines its unitary invariance group

$$*\langle g(v)|g(u)\rangle = *\langle v|u\rangle \Leftrightarrow g \in \mathbf{U}(d_+, d_-) \subset \mathbf{GL}(\mathbb{C}^d), \qquad d = d_+ + d_- \tag{3.2}$$

the d different classes of conjugations are characterized by the signatures (d_+, d_-).

With a fixed conjugation of $V \cong \mathbb{C}^d$, e.g., a Euclidean $\mathbf{U}(d)$ conjugation $\star$, given with a dual (V, V^T)-basis by $\star$: $e^A \leftrightarrow \delta^{AB}\check{e}_B$, any conjugation * is characterizable by a linear V-automorphism $\star \circ * \in \mathbf{GL}(\mathbb{C}^d)$.

3.2. The Indefinite Unitary Poincaré Group

The unitary conformal group $\mathbf{U}(n, n)$ and its Lie algebra $\mathbf{u}(n, n)$ for $n \geq 1$ can be illustrated in a complex $(n + n) \times (n + n)$ matrix block representation. A positive $\mathbf{U}(n)$ conjugation $\star$ defines an indefinite $\mathbf{U}(n, n)$ conjugation $\times$ via the automorphism

$$\star \circ \times \cong \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$$

We have

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow F^{\times} = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}\begin{pmatrix} a^\star & c^\star \\ b^\star & d^\star \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} = \begin{pmatrix} d^\star & b^\star \\ c^\star & a^\star \end{pmatrix}$$

$$\mathbf{U}(n, n) = \{G \in \mathbf{GL}(\mathbb{C}^{2n}) | G^{\times} = G^{-1}\} \tag{3.3}$$

$$\mathbf{u}(n, n) = \{L | L^{\times} = -L\}$$

$\mathbf{U}(n, n)$ contains a $\mathbf{GL}(\mathbb{C}^n)_{\mathbb{R}}$-isomorphic subgroup with its $\times$-antisymmetric Lie algebra $\mathbf{gl}(\mathbb{C}^n)_{\mathbb{R}}$ as a real $2n^2$-dimensional Lie symmetry,

$$\mathbf{GL}(\mathbb{C}^n)_{\mathbb{R}} \cong \mathbf{GL}(\mathbb{C}^n_2)_{\mathbb{R}} = \left\{G = \begin{pmatrix} g & 0 \\ 0 & g^{-1\star} \end{pmatrix}\right\}$$

$$\mathbf{GL}(\mathbb{C}^n)_{\mathbb{R}} = \mathbf{UL}(\mathbb{C}^n)_{\mathbb{R}} \times \mathrm{D}(1_n), \qquad \mathbf{UL}(\mathbb{C}^n)_{\mathbb{R}} = \mathbf{U}(1_n) \circ \mathbf{SL}(\mathbb{C}^n)_{\mathbb{R}} \tag{3.4}$$

$$\mathbf{gl}(\mathbb{C}^n)_{\mathbb{R}} \cong \mathbf{gl}(\mathbb{C}^n_2)_{\mathbb{R}} = \left\{L = \begin{pmatrix} l & 0 \\ 0 & -l^\star \end{pmatrix}\right\}$$

$$\mathbf{gl}(\mathbb{C}^n)_{\mathbb{R}} = \mathbf{u}(1_n) \oplus \mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}} \oplus \mathbf{d}(1_n) \cong \mathbb{R}^{2n^2}$$

The real Abelian Lie algebras involved are $\mathbf{u}(1_n) \cong \mathbb{R}$ for the phases and $\mathbf{d}(1_n) \cong \mathbb{R}$ for the dilatations. The remaining simple Lie algebra of rank $2(n - 1)$ is the generalized Lorentz Lie algebra $\mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}} \cong \mathbb{R}^{2(n^2-1)}$ containing the compact $\mathbf{SU}(n)$-Lie algebra

$$\mathbf{u}(1) \cong \mathbf{u}(1_{2n}) = \mathbb{R}\begin{pmatrix} i\mathbf{1}_n & 0 \\ 0 & i\mathbf{1}_n \end{pmatrix}, \qquad \mathbf{d}(1) \cong \mathbf{d}(1_n)_3 = \mathbb{R}\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}$$

$$\mathbf{sl}(\mathbb{C}^2)_{\mathbb{R}} \cong \mathbf{sl}(\mathbb{C}^n_2)_{\mathbb{R}} = \left\{\begin{pmatrix} l & 0 \\ 0 & -l^\star \end{pmatrix} \middle| \operatorname{tr} l = 0\right\} \cong \mathbb{R}^{2(n^2-1)} \tag{3.5}$$

$$\mathbf{su}(n) \cong \mathbf{su}(n_2) = \left\{\begin{pmatrix} il & 0 \\ 0 & il \end{pmatrix} \middle| \operatorname{tr} l = 0, l = l^\star\right\} \cong \mathbb{R}^{n^2-1}$$

A possible basis for the Lie algebra $\mathbf{sl}(\mathbb{C}^n)$ uses the $(n^2 - 1)$ generalized traceless Pauli, Gell-Mann, etc., matrices $\vec{\sigma}_n = \vec{\sigma}^\star_n$, nontrivial for $n \geq 2$,

$$\begin{pmatrix} i\vec{\sigma}_n & 0 \\ 0 & i\vec{\sigma}_n \end{pmatrix}, \qquad \begin{pmatrix} \vec{\sigma}_n & 0 \\ 0 & -\vec{\sigma}_n \end{pmatrix} \tag{3.6}$$

The real Lie algebra $\mathbf{su}(n, n)$ contains in addition a translation Lie algebra $\mathbf{t}(n^2)$ as a maximal Abelian ideal

$$\mathbf{t}(n^2) = \left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| x = -x^\star\right\} \cong \mathbb{R}^{n^2}, \qquad \text{basis: } \begin{pmatrix} 0 & i\mathbf{1}_n, i\vec{\sigma}_n \\ 0 & 0 \end{pmatrix} \tag{3.7}$$

The translations as a semidirect factor together with the phase, dilatations, and Lorentz transformations constitute the generalized indefinite unitary Poincaré Lie algebra

$$\mathbf{u}(n, n) \supset \mathbf{poinc}(n) \cong \mathbf{u}(1) \oplus \mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}} \oplus \mathbf{d}(1) \oplus \mathbf{t}(n^2) \cong \mathbb{R}^{3n^2}$$

$$\text{with} \quad \begin{cases} [\mathbf{u}(1), \mathbf{u}(1) \oplus \mathbf{sl}(\mathbb{C}^n) \oplus \mathbf{d}(1) \oplus \mathbf{t}(n^2)] = \{0\} \\ [\mathbf{d}(1), \mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}} \oplus \mathbf{d}(1)] = \{0\}, \qquad [\mathbf{d}(1), \mathbf{t}(n^2)] = \mathbf{t}(n^2) \\ [\mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}}, \mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}}] = \mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}} \\ [\mathbf{sl}(\mathbb{C}^n)_{\mathbb{R}}, \mathbf{t}(n^2)] = \mathbf{t}(n^2) \\ [\mathbf{t}(n^2), \mathbf{t}(n^2)] = \{0\} \end{cases} \tag{3.8}$$

3.3. Unitary Poincaré Groups for Time and Time-Space

For the generalized unitary Poincaré groups in the unitary conformal groups, the case $n = 1$, called unitary Poincaré group for time

$$\mathbf{u}(1, 1) \supset \mathbf{poinc}(1) \cong \mathbf{u}(1) \oplus \mathbf{d}(1) \oplus \mathbf{t}(1) \cong \mathbb{R}^3, \qquad \mathbf{t}(1) \cong \mathbb{R} \tag{3.9}$$

and the case $n = 2$, called unitary Poincaré group for Minkowski time-space

$$\mathbf{u}(2, 2) \supset \mathbf{poinc}(2) \cong \mathbf{u}(1) \oplus \mathbf{d}(1) \oplus \mathbf{sl}(\mathbb{C}^2)_{\mathbb{R}} \oplus \mathbf{t}(4) \cong \mathbb{R}^{12}, \qquad \mathbf{t}(4) \cong \mathbb{R}^4 \tag{3.10}$$

are distinguished. Only for $n = 1, 2$ do the defining complex n-dimensional representations of $\mathbf{SL}(\mathbb{C}^n)$ have an invariant bilinear form and, therewith, a bilinear form on the translation time $\mathbb{R}$ and time-space $\mathbb{R}^4$.

For $n = 1$ (time) with the trivial group $\mathbf{SL}(\mathbb{C}^1) = \{1\}$ the bilinear form is simply the product of two numbers, which induces a definite product

$$n = 1: \qquad \mathbf{t}(1) \ni t, s \mapsto ts \in \mathbb{R}, \quad t^2 \geq 0 \tag{3.11}$$

For $n = 2$ (time-space) the $\mathbf{SL}(\mathbb{C}^2)$-invariant totally antisymmetric spinor 'metric' $\epsilon^{AB} = -\epsilon^{BA}$ induces the Lorentz 'metric' g on Minkowski time-space, indefinite with signature (1, 3)

$$n = 2: \qquad \mathbf{t}(4) \ni x, y \mapsto g(x, y) = g(y, x) \in \mathbb{R}, \quad \text{sign}\, g = (1, 3) \tag{3.12}$$

3.4. Modality Groups

Any representation of the totally ordered additive group $(\mathbb{R}, +)$, called the causal group, in a unitary group, called the modality group, on a complex space $V \cong \mathbb{C}^d$, $d = d_+ + d_-$,

$$D: \quad \mathbb{R} \to \mathbf{U}(d_+, d_-), \quad \tau \mapsto D(\tau) \tag{3.13}$$

has a conjugation $*$ which implements the inversion of the causal group

$$D(\tau)^* = D(-\tau) \tag{3.14}$$

Any unitary causal group representation is built by nondecomposable ones. The nondecomposable representations of the causal group (Boerner, 1955; Saller, 1989) are characterized by an invariant $\mu \in \mathbb{R}$ and a dimension $d \in \mathbb{N}$. They are generated by iH_d, with H_d being the sum of the identity $\mathbf{1}_d$ on the representation space $V \cong \mathbb{C}^d$ and a power-d nilpotent element N_d

$$D_d(\cdot \mid \mu)\colon \quad \mathbb{R} \to \mathbf{U}_d(\mathbb{R}) \subset \mathbf{GL}(\mathbb{C}^d), \quad \begin{cases} D_d(\tau \mid \mu) = e^{i\tau H_d} \\ H_d = \mu \mathbf{1}_d + N_d \\ \text{for } d = 1\colon \quad N_1 = 0 \\ \text{for } d \geq 2\colon \quad \begin{cases} (N_d)^{d-1} \neq 0 \\ (N_d)^d = 0 \end{cases} \end{cases} \tag{3.15}$$

The modality groups of the nondecomposable representations are given by

$$\mathbf{U}_d(\mathbb{R}) = \begin{cases} \mathbf{U}\left(\frac{d+1}{2}, \frac{d-1}{2}\right) & \text{for} \quad d = 1, 3, \ldots \\ \mathbf{U}\left(\frac{d}{2}, \frac{d}{2}\right) & \text{for} \quad d = 2, 4, \ldots \end{cases} \tag{3.16}$$

Only the $\mathbf{U}(1)$-representations (Fourier representations) of the causal group $\mathbb{R}$ are irreducible and positive unitary; they are not faithful. We have

$$D_1(\tau \mid \mu) = e^{i\tau\mu} = D_1(-\tau \mid \mu)^\star \in \mathbf{U}(1) \subset \mathbf{GL}(\mathbb{C}) \tag{3.17}$$

The lowest dimensional injective representations are the indefinite unitary reducible, but nondecomposable $d = 2$ representations

$$D_2(\tau \mid \mu) = e^{i\tau\mu}\begin{pmatrix} 1 & i\tau \\ 0 & 1 \end{pmatrix} = D_2(-\tau \mid \mu)^\times = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_2(\tau \mid \mu)^\star \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\in \mathbf{U}(1, 1) \subset \mathbf{GL}(\mathbb{C}^2) \tag{3.18}$$

Their antisymmetric twofold product gives the irreducible $\mathbf{U}(1)$-representation $D_1(\tau \mid 2\mu)$; their totally symmetric products give all nondecomposable, indefinite, unitary faithful $\mathbf{U}_d(\mathbb{R})$-representations $D_d(\tau \mid (d-1)\mu)$, $d = 2, 3, \ldots$.

3.5. Modality Groups for Translations

The additive group of translations $\mathbb{R}^{n^2}$ has the irreducible, nonfaithful Fourier representations in the positive unitary modality group $\mathbf{U}(1)$

$$D_1(\cdot\,|q):\quad \mathbb{R}^{n^2} \to \mathbf{U}(1),\quad D_1(x|q) = e^{i\langle x,q\rangle} \tag{3.19}$$

characterized by a linear form q ('energy-momenta') of the translations.

Faithful representations are possible in the subgroups $\mathbf{U}(1_{2n}) \times \mathbf{T}(n^2)$ of the unitary Poincaré groups with the translation group $\mathbf{T}(n^2) = \exp[\mathbf{t}(n^2)]$,

$$D_2(\cdot\,|q):\quad \mathbb{R}^{n^2} \to \mathbf{U}(1_{2n}) \times \mathbf{T}(n^2) \subset \mathbf{U}(n, n)$$

$$D_2(x|q) = e^{i\langle x,q\rangle}\begin{pmatrix} \mathbf{1}_n & i(x_0\mathbf{1}_n + \vec{x}\vec{\sigma}_n) \\ 0 & \mathbf{1}_n \end{pmatrix} \tag{3.20}$$

Those representations have the indefinite modality group $\mathbf{U}(n, n)$.

3.6. Quantum Algebras and Quantum Invariants

Any complex vector space $V \cong \mathbb{C}^d$ defines its quantum algebra (Saller, 1993a,c) $\mathbf{Q}_\epsilon(\mathbb{C}^{2d})$ of Fermi or Bose type $\epsilon = \pm 1$ as a Clifford algebra over the direct sum space $\mathbf{V} = V \oplus V^T \cong \mathbb{C}^{2d}$ with the linear forms V^T. The Clifford factorization of the tensor algebra $\otimes\mathbf{V}$ is performed with the dual product, extended ϵ-symmetrically as bilinear form of $\mathbf{V}$, leading to the characteristic Fermi and Bose (anti)commutators, given in a dual basis $\{e^A, \check{e}_B\}_{A,B=1}^d$ of (V, V^T) by

$$\text{in}\quad \mathbf{Q}_\epsilon(\mathbb{C}^{2d}),\quad \epsilon = \pm 1:\quad \begin{cases} [\check{e}_A, e^B]_\epsilon = \delta_A^B \\ [\check{e}_A, \check{e}_B]_\epsilon = 0 = [e^A, e^B]_\epsilon = 0 \end{cases} \tag{3.21}$$

The Lie algebra of the basic space endomorphisms is represented by inner derivations of the quantum algebras.

The quantum algebra functors $\mathbf{Q}_\epsilon$ are exponential, i.e., the quantum algebra of a direct sum space $V \cong V_1 \oplus V_2$ is isomorphic to the tensor product of the individual quantum algebras

$$\mathbf{Q}_\epsilon(\mathbf{V}_1 \oplus \mathbf{V}_2) \cong \mathbf{Q}_\epsilon(\mathbf{V}_1) \otimes \mathbf{Q}_\epsilon(\mathbf{V}_2) \tag{3.22}$$

The $\mathbf{GL}(\mathbb{C}^n)_\mathbb{R}$-invariants $\mathbb{C}[I]$ in a quantum algebra $\mathbf{Q}_\epsilon(\mathbb{C}^{2d})$ will be defined to be those quantum elements which commute with all endomorphisms of the basic vector space $V \cong \mathbb{C}^d$,

$$\mathbb{C}[I] = \{a \in \mathbf{Q}_\epsilon(\mathbb{C}^{2d}) \,|\, [e^A\check{e}_B, a] = 0 \text{ for all } A, B = 1, \ldots, d\} \tag{3.23}$$

They are generated by the basic space identity or by

$$I = \frac{[e^A, \check{e}_A]_{-\epsilon}}{2} = e^A \check{e}_A - \epsilon \frac{d}{2} = -\epsilon \check{e}_A e^A + \epsilon \frac{d}{2} \tag{3.24}$$

Bose quantum algebras $\mathbf{Q}_-(\mathbb{C}^{2d})$ have countably infinite complex dimension $\aleph_0$. In this case the identity I is transcendental in the quantum algebra and the ring of invariants $\mathbb{C}[I]$ is isomorphic to the complex polynomials in one indeterminate.

For Fermi quantum algebras which are, because of the nilquadratic basic vectors (Pauli's principle), e.g., $e^1 e^1 = 0$, finite dimensional $\mathbf{Q}_+(\mathbb{C}^{2d}) \cong \mathbb{C}^{4^d}$, the identity I is algebraic in the quantum algebra

$$\text{in } \mathbf{Q}_+(\mathbb{C}^{2d}): \quad \left(I - \frac{d}{2}\right)\left(I - \frac{d}{2} + 1\right) \cdots \left(I + \frac{d}{2} - 1\right)\left(I + \frac{d}{2}\right) = 0 \tag{3.25}$$

Therefore the I-polynomials $\mathbb{C}[I]$ have maximal degree d.

3.7. Causal Quantum Modalities

A complex representation of the causal group $(\mathbb{R}, +)$ on a complex vector space $V \cong \mathbb{C}^d$ with basis $\{e^A\}$ and dual basis $\{\check{e}_A\}$ can be canonically extended to the quantum algebras $\mathbf{Q}_\epsilon(\mathbb{C}^{2d})$ for the representation space $V \oplus V^T \cong \mathbb{C}^{2d}$. The modality group $\mathbf{U}(d_+, d_-)$ of the causal group representation determines a conjugation of the quantum algebra.

The generator $iI(\mathrm{u})$ for a positive-definite $\mathbf{U}(1)$ representation of the causal group $\mathbb{R}$ on the space $V \cong \mathbb{C}$ is given in the quantum algebras as follows:

$$\mathbf{Q}_\epsilon(\mathbb{C}^2) \text{ with conjugation } \star \text{ of } \mathbf{U}(1): \quad \begin{cases} e = \mathrm{u}, \quad \check{e} = \mathrm{u}^\star \\ [\mathrm{u}^\star, \mathrm{u}]_\epsilon = 1 \\ [\mathrm{u}^\star, \mathrm{u}^\star]_\epsilon = 0 = [\mathrm{u}, \mathrm{u}]_\epsilon \end{cases} \tag{3.26}$$

$$I(\mathrm{u}) = \mu \frac{[e, \check{e}]_{-\epsilon}}{2} = \mu \frac{[\mathrm{u}, \mathrm{u}^\star]_{-\epsilon}}{2}$$

The generator $iH(\mathrm{b}, \mathrm{g})$ for an indefinite $\mathbf{U}(1, 1)$ representation of the causal group $\mathbb{R}$ on the space $V \cong \mathbb{C}^2$ with its semisimple and nilpotent part $I(\mathrm{b}, \mathrm{g})$ and $N(\mathrm{g})$, resp., is given in the quantum algebras as follows:

$$\mathbf{Q}_\epsilon(\mathbb{C}^4) \text{ with conjugation } \times \text{ of } \mathbf{U}(1, 1): \quad \begin{cases} e^1 = \mathrm{g}, \quad e^2 = \mathrm{b} \\ \check{e}_1 = \mathrm{b}^\times, \quad \check{e}_2 = \mathrm{g}^\times \\ [\mathrm{g}^\times, \mathrm{b}]_\epsilon = 1 = [\mathrm{b}^\times, \mathrm{g}]_\epsilon \\ [\mathrm{g}^\times, \mathrm{g}]_\epsilon = 0 = [\mathrm{b}^\times, \mathrm{b}]_\epsilon \\ \text{etc.} \end{cases}$$

$$H(\mathrm{b}, \mathrm{g}) = \mu \frac{[\mathrm{g}, \mathrm{b}^\times]_{-\epsilon} + [\mathrm{b}, \mathrm{g}^\times]_{-\epsilon}}{2} + \mathrm{g}\mathrm{g}^\times = I(\mathrm{b}, \mathrm{g}) + N(\mathrm{g}) \tag{3.27}$$

The eigenvectors of the basic space are denoted by g ('good'), the nilvectors, i.e., principal vectors which are not eigenvectors, by b ('bad').

The quantized $\mathbf{U}(n, n)$ representations of the translations $\mathbf{R}^{n^2}$ in the quantum algebras $\mathbf{Q}_\epsilon(\mathbf{C}^{4n})$ have n^2 generators $iQ(\mathrm{b}, \mathrm{g})^j$,

$$\mathbf{Q}_\epsilon(\mathbf{C}^{4n}) \text{ with conjugation } \times \text{ of } \mathbf{U}(n, n): \quad \begin{cases} \mathrm{g}^A, \mathrm{b}^A, \quad A = 1, \ldots, n \\ \mathrm{b}_A^\times, \mathrm{g}_A^\times \\ [\mathrm{g}_A^\times, \mathrm{b}^B]_\epsilon = \delta_A^B = [\mathrm{b}_A^\times, \mathrm{g}^B]_\epsilon \\ [\mathrm{g}_A^\times, \mathrm{g}^B]_\epsilon = 0 = [\mathrm{b}_A^\times, \mathrm{b}^B]_\epsilon \\ \text{etc.} \end{cases}$$

$$Q(\mathrm{b}, \mathrm{g})^j = q^j \frac{[\mathrm{g}^A, \mathrm{b}_A^\times]_{-\epsilon} + [\mathrm{b}^A, \mathrm{g}_A^\times]_{-\epsilon}}{2} + (\rho^j)_{AB}^{\;B} \mathrm{g}^A \mathrm{g}_B^\times = q^j I(\mathrm{b}, \mathrm{g}) + N(\mathrm{g})^j$$
$$\text{with } \rho^j \cong (\mathbf{1}_n, \vec{\sigma}_n) \tag{3.28}$$

In spaces with reducible, but nondecomposable representations of the causal group (R, +), the eigenvectors for the translations form a true subspace of all vectors with the action of the causal group.

In quantum algebras with a causal group representation on the basic space $V \cong \mathbf{C}^d$, the subalgebra for the eigenvectors of the translations is given by the invariants of the nilpotent part N of the generator $H = I + N$

$$\textbf{eigen } \mathbf{Q}_\epsilon(\mathbf{C}^{2d}) = \{a \in \mathbf{Q}_\epsilon(\mathbf{C}^{2d}) \,|\, [N_d, a] = 0\} \tag{3.29}$$

Obviously for $\mathbf{U}(1)$-modality in the quantum algebras $\mathbf{Q}_\epsilon(\mathbf{C}^2)$, the subalgebra for the eigenvectors is the full algebra

$$d = 1: \quad N_1 = 0 \Rightarrow \textbf{eigen } \mathbf{Q}_\epsilon(\mathbf{C}^2) = \mathbf{Q}_\epsilon(\mathbf{C}^2) \tag{3.30}$$

For $\mathbf{U}(1, 1)$-modality the subalgebra for the eigenvectors is a true subalgebra generated by the basic space eigenvectors g, $\mathrm{g}^\times$ and the basic space identity

$$d = 2: \quad \left\{1, \mathrm{g}, \mathrm{g}^\times, I(\mathrm{b}, \mathrm{g}) = \frac{[\mathrm{g}, \mathrm{b}^\times]_{-\epsilon} + [\mathrm{b}, \mathrm{g}^\times]_{-\epsilon}}{2}, [\mathrm{b}, \mathrm{g}], [\mathrm{g}^\times, \mathrm{b}^\times]\right\}$$
$$\text{generates } \textbf{eigen } \mathbf{Q}_\epsilon(\mathbf{C}^4) \tag{3.31}$$

The commutators [b, g] and $[\mathrm{g}^\times, \mathrm{b}^\times]$ are nontrivial only in the Fermi quantum algebra.

For $\mathbf{U}(n, n)$, $n \geq 2$, one obtains as generating system

$$d = 2n: \quad \left\{1, g^A, g_A^{\times}, I(b, g) = \frac{[g^A, b_A^{\times}]_{-\epsilon} + [b^A, g_A^{\times}]_{-\epsilon}}{2} \middle| A = 1, \ldots, n\right\}$$

$$\text{generates } \mathbf{eigen}\ \mathbf{Q}_\epsilon(\mathbf{C}^{4n}) \tag{3.32}$$

3.8. Fock and Heisenberg Forms of Quantum Algebras

Expectation values for quantum elements arise with linear quantum algebra forms (Saller, 1992a). Such forms will be required to be invariant with respect to the adjoint action of the basic space endomorphisms, i.e., they can be nontrivial only on the ring of quantum invariants $\mathbf{C}[I]$, generated by the identity $I = ([\check{e}_A, e^A]_{-\epsilon})/2$

$$\langle \cdot \rangle_d: \quad \mathbf{Q}_\epsilon(\mathbf{C}^{2d}) \to \mathbf{C}, \quad a \mapsto \langle a \rangle_d \tag{3.33}$$

$$a \notin \mathbf{C}[I] \Rightarrow \langle a \rangle_d = 0$$

Since the ring of invariants is Abelian, quantum algebra forms will be required to be Abelian thereon. Therefore they are completely determined by the form value $\langle I \rangle_d$ of the generating invariant I

$$\langle I^k \rangle_d = (\langle I \rangle_d)^k, \qquad k = 0, 1, \ldots \tag{3.34}$$

In Fermi quantum algebras $\mathbf{Q}_+(\mathbf{C}^{2d})$ the identity I is algebraic of degree d. Therefore its form value can be only one of the zeros of the minimal I-polynomial

$$\text{in} \quad \mathbf{Q}_+(\mathbf{C}^{2d}): \quad \langle I \rangle_d = \frac{d}{2}, \frac{d}{2} - 1, \ldots, 1 - \frac{d}{2}, -\frac{d}{2} \tag{3.35}$$

Since a quantum algebra $\mathbf{Q}_\epsilon(\mathbf{C}^{2d})$ of a vector space V is isomorphic to the tensor product of its factors with respect to a direct sum $V \cong V_1 \oplus V_2$, where $V_{1,2}$ carry nondecomposable causal group representations, a linear form is required to be writable as a product form on the corresponding quantum algebra factors

$$\mathbf{Q}_\epsilon(\mathbf{V}_1 \oplus \mathbf{V}_2) \cong \mathbf{Q}_\epsilon(\mathbf{V}_1) \otimes \mathbf{Q}_\epsilon(\mathbf{V}_2) \Rightarrow \langle \cdot \rangle_d = \langle \cdot \rangle_{d_1} \langle \cdot \rangle_{d_2} \tag{3.36}$$

$$a = a_1 a_2 \Rightarrow \langle a \rangle_d = \langle a_1 \rangle_{d_1} \langle a_2 \rangle_{d_2}$$

If there occur only irreducible causal group representations, the possible forms of the 'smallest' quantum algebras $\mathbf{Q}_\epsilon(\mathbf{C}^2)$ determine all quantum algebra forms. For the irreducible representations $D_1(\tau \mid \mu)$ of the causal group on $V \cong \mathbb{C}$, the nonfactorizable $\mathbf{Q}_\epsilon(\mathbb{C}^2)$-forms are determined by the possible form values $\langle I \rangle_1$ of the identity I,

$$\langle \cdot \rangle_1: \quad \mathbf{Q}_\epsilon(\mathbb{C}^2) \to \mathbb{C} \text{ determined by } \begin{cases} [\check{e}, e]_\epsilon = 1 \\ \left\langle \dfrac{[e, \check{e}]_{-\epsilon}}{2} \right\rangle_1 = \langle I \rangle_1 \end{cases} \tag{3.37}$$

$$\Rightarrow \langle \check{e}e \rangle_1 = \frac{1 - 2\epsilon\langle I \rangle_1}{2} \quad \text{and} \quad \epsilon\langle e\check{e} \rangle_1 = \frac{1 + 2\epsilon\langle I \rangle_1}{2}$$

For Fermi quantum algebras $\mathbf{Q}_+(\mathbb{C}^2)$ there are only two forms, determined by $\langle I \rangle_1 = \mp 1/2$, which trivialize one of the forms $\langle e\check{e} \rangle_1$ or $\langle \check{e}e \rangle_1$. This structure is taken over also for the Bose case

$$\epsilon\langle I \rangle_1 = \epsilon\left\langle \frac{[e, \check{e}]_{-\epsilon}}{2} \right\rangle_1 = \mp\frac{1}{2} \Rightarrow \begin{cases} \langle \check{e}e \rangle_1 = 1 \text{ and } \epsilon\langle e\check{e} \rangle_1 = 0 \\ \langle \check{e}e \rangle_1 = 0 \text{ and } \epsilon\langle e\check{e} \rangle_1 = 1 \end{cases} \tag{3.38}$$

$$\text{U(1)-conjugation:} \quad e = u, \quad \check{e} = \begin{cases} u^\star \text{ for } \epsilon\langle I \rangle_1 = -1/2 \\ \epsilon u^\star \text{ for } \epsilon\langle I \rangle_1 = 1/2 \end{cases}$$

With those two nonfactorizable forms on the quantum algebras $\mathbf{Q}_\epsilon(\mathbb{C}^2)$ over a 1-dimensional space $V \cong \mathbb{C}$ with an irreducible causal group representation, factorizable forms of $\mathbf{Q}_\epsilon(\mathbb{C}^{2d})$ with signature (d_+, d_-) for $[\mathrm{U}(1)]^{d_+} \times [\mathrm{U}(1)]^{d_-}$ can be combined

$$\text{Fock forms of } \mathbf{Q}_\epsilon(\mathbb{C}^{2d}) \cong \overset{d}{\otimes} \mathbf{Q}_\epsilon(\mathbb{C}^2): \quad \begin{cases} \epsilon\langle I \rangle_d = \dfrac{d_+ - d_-}{2} \\ \text{for } d_+ + d_- = d = 1, 2, \ldots \end{cases} \tag{3.39}$$

Fock forms come with the distinction of a basis $\{u^A\}_{A=1}^d$ and a decomposition $V = \bigoplus_{A=1}^d \mathbb{C}u^A \cong \mathbb{C}^d$ into irreducible 1-dimensional representation spaces for the causal group. They can also be called Sylvester forms or oscillator forms or Abelian forms.

Fermi quantum algebras $\mathbf{Q}_+(\mathbb{C}^{2d})$, in contrast to Bose quantum algebras, have a linear reflection between the basic vectors V and linear forms V^T which keeps invariant the quantization, but inverts the identity I:

$$e^A \leftrightarrow \check{e}_A: \quad \begin{cases} \{\check{e}_A, e^A\} \leftrightarrow \{e^A, \check{e}_A\} \quad \text{(invariant)} \\ I = \dfrac{e^A\check{e}_A - \check{e}^A e_A}{2} \leftrightarrow -I, \quad \langle I \rangle_d \leftrightarrow \langle -I \rangle_d \end{cases} \tag{3.40}$$

The forms of Fermi quantum algebras over vector spaces with even dimension d allow a reflection-compatible trivial form value

$$\text{on } \mathbf{Q}_+(\mathbb{C}^{2d}): \quad \langle I \rangle_d = 0 \quad \text{for } d = 2, 4, \ldots \tag{3.41}$$

Such forms can be combined by forms with $\langle I\rangle_2 = 0$ on $\mathbf{Q}_+(\mathbf{C}^4)$ over a 2-dimensional vector space $V \cong \mathbf{C}^2$ with a faithful nondecomposable representation $D_2(\tau\,|\,\mu)$ of the causal group and an indefinite $\mathbf{U}(1, 1)$-conjugation

$$\langle\cdot\rangle_2:\quad \mathbf{Q}_+(\mathbf{C}^4) \mapsto \mathbf{C} \quad \text{determined by} \quad \begin{cases} \{g^\times, b\} = 1 = \{b^\times, g\} \\ \langle[g^\times, b]\rangle_2 = 0 = \langle[b^\times, g]\rangle_2 \end{cases}$$

$$\Rightarrow \langle g^\times b\rangle_2 = \langle bg^\times\rangle_2 = \langle b^\times g\rangle_2 = \langle gb^\times\rangle_2 = \frac{1}{2} \qquad (3.42)$$

The combined forms have signature $(d/2, d/2)$ for $[\mathbf{U}(1, 1)]^{d/2} \subseteq \mathbf{U}(d/2, d/2)$:

$$\text{Heisenberg forms of } \mathbf{Q}_+(\mathbf{C}^{2d}) \cong \bigotimes^{d/2} \mathbf{Q}_+(\mathbf{C}^4):\quad \begin{cases} \langle I\rangle_d = 0 \\ \text{for } d = 2, 4, \ldots \end{cases} \qquad (3.43)$$

Heisenberg forms come with the distinction of a 'pair' basis $\{g^A, b^A\}_{A=1}^{d/2}$ and a decomposition $V \cong \bigoplus_{A=1}^{d/2} (\mathbf{C}g^A + \mathbf{C}b^A) \cong \mathbf{C}^d$ into nondecomposable 2-dimensional representation spaces for the causal group. They can also be called Witt forms or non-Abelian forms.

3.9. Quantum Algebras with Inner Products

With both a conjugation $*$ from the basic space $V \cong \mathbf{C}^d$ induced on a quantum algebra $\mathbf{Q}_\epsilon(\mathbf{C}^{2d})$ and a linear quantum algebra form $\langle\cdot\rangle_d$, which is conjugation-compatible $\langle a^*\rangle_d = \overline{\langle a\rangle_d}$, the quantum algebra carries an inner product

$$*\langle\cdot\,|\,\cdot\rangle:\quad \mathbf{Q}_\epsilon(\mathbf{C}^{2d}) \times \mathbf{Q}_\epsilon(\mathbf{C}^{2d}) \to \mathbf{C}, \qquad *\langle a\,|\,b\rangle = \langle a^*b\rangle_d = \overline{*\langle b\,|\,a\rangle} \qquad (3.44)$$

The invariance group $\mathbf{U}(d_+, d_-)$ for the conjugation $*$ of the basic space $V \cong \mathbf{C}^d$ determines the positive or indefinite structure of the inner product of the quantum algebra.

The factorization of a quantum algebra with the left ideal of the orthogonal for the inner product (Gelfand–Naimark–Segal construction)

$$\mathbf{Q}_\epsilon(\mathbf{C}^{2d})^\perp = \{n \in \mathbf{Q}_\epsilon(\mathbf{C}^{2d}) \,|\, \langle a^*n\rangle_d = *\langle a\,|\,n\rangle = 0 \text{ for all } a \in \mathbf{Q}_\epsilon(\mathbf{C}^{2d})\} \qquad (3.45)$$

determines the vector space $\mathbf{Q}_\epsilon(\mathbf{C}^{2d})/\mathbf{Q}_\epsilon(\mathbf{C}^{2d})^\perp$ where the classes carry an induced nondegenerate inner product.

4. UNITARIZATION FOR INDEFINITE METRIC FIELDS

Quantum fields describe both particles and interactions. An experimenter in a laboratory uses an asymptotic space spanned by Wigner particle states, which has to be interpretable with probabilities.

A free relativistic particle quantum field $\mathbf{\Phi}(x|m)$ with mass $m \geq 0$, Fermi or Bose $\epsilon = \pm 1$, is characterized by its spacelike trivial quantization distribution $\mathbf{s}(x|m)$ (principal value integration m_P^2 in the energy plane)

$$[\mathbf{\Phi}, \mathbf{\Phi}]_\epsilon(x|m) = [\mathbf{\Phi}(0|m), \mathbf{\Phi}(x|m)]_\epsilon = i\mathbf{s}(x|m), \frac{\partial_k}{m}\mathbf{s}(x|m), \ldots$$

$$= 0 \text{ for } x^2 < 0 \tag{4.1}$$

$$i\mathbf{s}(x|m) = \int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\,\epsilon(q_0)\delta(m^2 - q^2) = \frac{i\epsilon(x_0)}{\pi}\int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\frac{1}{m_P^2 - q^2}$$

and its expectation function for the 'opposite' commutator $\mathbf{C}(x|m)$, which is supported also spacelike

$$\langle[\mathbf{\Phi}, \mathbf{\Phi}]_{-\epsilon}\rangle(x|m) = \langle[\mathbf{\Phi}(0|m), \mathbf{\Phi}(x|m)]_{-\epsilon}\rangle = \mathbf{C}(x|m), -\frac{i\partial_k}{m}\mathbf{C}(x|m), \ldots$$

$$\mathbf{C}(x|m) = \int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\,\delta(m^2 - q^2) \tag{4.2}$$

The expectation function $\mathbf{C}$—not the causally supported quantization distribution $\mathbf{s}$—relies on the metrical structure of the quantum fields with respect to the inner product induced by both a linear quantum algebra form and a conjugation (Section 3), connected with the time-space translation representations.

The sum of causally ordered quantization distribution and expectation function is the Feynman propagator

$$\langle\mathcal{T}\mathbf{\Phi}\mathbf{\Phi}\rangle(x|-im) = -\epsilon(x_0)[\mathbf{\Phi}, \mathbf{\Phi}]_\epsilon(x|m) + \langle[\mathbf{\Phi}, \mathbf{\Phi}]_{-\epsilon}\rangle(x|m)$$

$$= \mathbf{f}(x|-im), -\frac{i\partial_k}{m}\mathbf{f}(x|-im), \ldots \tag{4.3}$$

together with the conjugated distribution ('anti-Feynman propagator') given as follows:

$$\mathbf{f}(x|\pm im) = \pm i\epsilon(x_0)\mathbf{s}(x|m) + \mathbf{C}(x|m) = \overline{\mathbf{f}(x|\mp im)}$$

$$= \int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\,2\theta(\pm x_0q_0)\delta(m^2 - q^2) \tag{4.4}$$

$$= \pm\frac{i}{\pi}\int \frac{d^4q\, e^{ixq}}{(2\pi)^3}\frac{1}{m^2 \pm io - q^2} = \int \frac{d^3q\, e^{-i\vec{x}\vec{q}}}{(2\pi)^3 q_0}\, e^{\pm i|x_0|q_0}$$

The Feynman combinations $\mathbf{f}(x | \pm im)$ contain the sign functions $[1 \pm \epsilon(x_0 q_0)]/2 = \theta(\pm x_0 q_0)$, relating to each other the causal structures of time translations and energies; they allow nontrivial in- and outgoing states.

The time integrals of the Feynman distributions exhibit via the Yukawa potential the interaction structure realized by the relativistic quantum fields. They involve only the quantization distribution $\mathbf{s}(x | m)$ and are independent of the inner product structure

$$\mp i \int dx_0 \, \mathbf{f}(x | \pm im) = \int d|x_0| \, \mathbf{s}(x | m) = \frac{e^{-|\vec{x}|m}}{2\pi|\vec{x}|} \tag{4.5}$$

Here time and energy integration have been interchanged.

The space integral of the Feynman distributions gives a causally ordered time representation

$$\int d^3x \, \mathbf{f}(x | \pm im) = \frac{e^{\pm i|x_0|m}}{m} \tag{4.6}$$

Here space and momentum integration have been interchanged.

For time $x_0 = 0$ only the inner-product-dependent expectation function $\mathbf{C}(x | m)$ contributes nontrivially

$$\int d^3x \, \mathbf{f}(\vec{x} | \pm im) = \int d^3x \, \mathbf{C}(\vec{x} | m) = \frac{1}{m} \tag{4.7}$$

A quantum algebra for fields with an indefinite modality group $\mathbf{U}(n, n)$ carries an indefinite inner product (Section 3); it is in danger to lead via the expectation function to 'negative probabilities.' The dangerous quantum algebra elements with 'negative norm' are relevant for a local formulation of relativistic interactions, e.g., for the Coulomb interactions (Section 2.1). Since such fields have no particle interpretation and have to be avoided as in- and outgoing states, they should contribute only with their interaction describing quantization distributions.

The nilpotent part in the representation of the time-space translations provides a projection to cut out a subalgebra of time-space translation eigenvectors (particles). They carry a positive-definite inner product and build the asymptotic state space.

4.1. Unitarity for Particle Fields

The realization of the probabilistic structure for relativistic fields with a complete particle interpretation (Section 1) is simple: Such fields represent the time-space translations in the group **U**(1) or, more exactly in **U**(1_d) for d degrees of freedom, generated by $iI(\mathrm{u})$

$$[\mathrm{u}_\beta^\star, \mathrm{u}^\alpha]_\epsilon = \delta_\beta^\alpha, \qquad I(\mathrm{u}) = \sum_{\alpha=1}^{d} I(\mathrm{u}^\alpha) \qquad \text{with} \quad I(\mathrm{u}^\alpha) = \frac{[\mathrm{u}^\alpha, \mathrm{u}_\alpha^\star]_{-\epsilon}}{2} \tag{4.8}$$

For fields with momentum-dependent harmonic components $\mathrm{u}(\vec{q})$ one has to include a sum with $\int [d^3q/(2\pi)^3 q_0]$ (Section 1). The local stability group, e.g., spin **SU**(2) and **SO**(3) or circularity (polarization) **U**(1) and **SO**(2), has to be compatible with the modality group **U**(1).

The quantum algebra $\mathbf{Q}_\epsilon(\mathrm{C}^{2d})$ for the harmonic components u^α, $\mathrm{u}_\alpha^\star$ is the product of d individual quantum algebras $\mathbf{Q}_\epsilon(\mathrm{C}^2)$, one for each α. These factors carry via the Fock form $\langle\cdot\rangle_1$ and the **U**(1)-conjugation $\star$ a positive-definite inner product (Sections 3.8, 3.9), e.g., shown in an orthogonal $\mathbf{Q}_\epsilon(\mathrm{C}^2)$-basis $\{\mathrm{u}^k\mathrm{u}^{\star l} | k, l = 0, 1, \ldots\}$ (for Fermi algebras only $k, l = 0, 1$)

$$\text{for } \mathbf{Q}_\epsilon(\mathrm{C}^2){:} \quad \langle I(\mathrm{u})\rangle_1 = -\frac{\epsilon}{2} \Rightarrow \begin{cases} \langle(\mathrm{u}^\star\mathrm{u})^k\rangle_1 = 1 \\ \langle \mathrm{u}^{k\star}\mathrm{u}^l\rangle_1 = k!\delta_{kl} \\ {}^\star\langle \mathrm{u}^k\mathrm{u}^{\star m} | \mathrm{u}^l\mathrm{u}^{\star n}\rangle = k!\delta_{m0}\delta_{n0}\delta_{kl} \end{cases} \tag{4.9}$$

The asymptotic particle Fock space can be spanned by the classes of the norm nontrivial vectors $\{\mathrm{u}^k | k = 0, 1, \ldots\}$.

4.2. Unitarization for Gauge Fields

The dangerous indefinite structures for Maxwell–Witt fields $\mathbf{A}(x)^k$ (gauge fields) arise because of the representation of the translation group for the (0, 3)-degrees of freedom in the indefinite unitary group **U**(1, 1)—with the symbols of Section 2.1,

$$\text{in } \mathbf{Q}_-(\mathrm{C}^4){:} \quad [\mathrm{G}^\times, \mathrm{B}] = 1 = [\mathrm{B}^\times, \mathrm{G}] \tag{4.10}$$

$$H(\mathrm{B}, \mathrm{G}) = \frac{\{\mathrm{B}, \mathrm{G}^\times\} + \{\mathrm{G}, \mathrm{B}^\times\}}{2} + \frac{\mathrm{G}\mathrm{G}^\times}{M_0} = I(\mathrm{B}, \mathrm{G}) + N(\mathrm{G})$$

In contrast to G, $\mathrm{G}^\times$ ('good'), the vectors B, $\mathrm{B}^\times$ ('bad') are not eigenvectors of the time translations. They have to be avoided in the asymptotic particle space.

The Fock form $\langle\cdot\rangle_2$ with the **U**(1, 1)-conjugation $\times$ gives an indefinite inner product ${}^\times\langle a | b\rangle = \langle a^\times b\rangle_2$ of the Bose quantum algebra $\mathbf{Q}_-(\mathrm{C}^4)$,

$$\langle I(\mathrm{B}, \mathrm{G})\rangle_2 = 1 \Rightarrow \begin{cases} \langle \mathrm{B}^\times\mathrm{G}\rangle_2 = 1 = \langle \mathrm{G}^\times\mathrm{B}\rangle_2 \\ \langle \mathrm{G}^\times\mathrm{G}\rangle_2 = 0 = \langle \mathrm{B}^\times\mathrm{B}\rangle_2 \\ \left\langle \dfrac{\mathrm{G}^\times \pm \mathrm{B}^\times}{\sqrt{2}} \dfrac{\mathrm{G} \pm \mathrm{B}}{\sqrt{2}} \right\rangle_2 = {}^\times\left\langle \dfrac{\mathrm{G} \pm \mathrm{B}}{\sqrt{2}} \middle| \dfrac{\mathrm{G} \pm \mathrm{B}}{\sqrt{2}} \right\rangle = \pm 1 \end{cases} \tag{4.11}$$

Asymptotic help comes from the Fadeev–Popov fields (Section 2.2) which have a 'twin' structure with respect to the (0, 3)-components of the Maxwell–Witt fields

$$\text{in } \mathbf{Q}_+(C^4): \quad \{a^\times, u\} = 1 = \{u^\times, a\} \tag{4.12}$$
$$H(a, u) = \frac{[u, a^\times] + [a, u^\times]}{2} + \frac{uu^\times}{N_0} = I(a, u) + N(u)$$

They have an indefinite Fock inner product, too:

$$\langle I(a, u)\rangle_2 = -1 \Rightarrow \begin{cases} \langle a^\times u\rangle_2 = 1 = \langle u^\times a\rangle_2 \\ \langle u^\times u\rangle_2 = 0 = \langle a^\times a\rangle_2 \\ \left\langle \frac{u^\times \pm a^\times}{\sqrt{2}} \frac{u \pm a}{\sqrt{2}} \right\rangle_2 = {}_\times\!\left\langle \frac{u \pm a}{\sqrt{2}} \middle| \frac{u \pm a}{\sqrt{2}} \right\rangle = \pm 1 \end{cases} \tag{4.13}$$

The generator for the translation group representation for both the non-particle gauge field degrees of freedom and the Fadeev–Popov fields in $\mathbf{U}(1, 1) \times \mathbf{U}(1, 1)$

$$H(B, G, a, u) = H(B, G) + H(a, u) \tag{4.14}$$

is invariant under the Becchi–Rouet–Stora transformation (Becchi *et al.*, 1976), which—in a quantum framework—replaces the classical gauge transformation. The BRS transformation is effected by a nilquadratic Fermi element $N(G, u)$ in the product quantum algebra $\mathbf{Q}_-(C^4) \otimes \mathbf{Q}_+(C^4)$ which is compatible with the translation action (Saller, 1991, 1992b)

$$N(G, u) = i(Gu^\times - uG^\times),$$
$$N(G, u)^2 = 0,$$
$$[H(B, G, a, u), N(G, u)] = 0 \tag{4.15}$$

The BRS charge $N(G, u)$ acts by the hybrid (Z_2-graded) bracket $[\![N, a]\!]$ on the quantum elements, i.e., with a commutator on Bose and an anticommutator on Fermi elements.

Only the translation eigenfields $\mathbf{G}(x)$ (gauge-fixing Bose field) and $\mathbf{U}(x)^k_+$ (Fadeev–Popov Fermi field) can be combined to a nilpotent Lorentz vector current $\mathbf{N}(x)^j$ in a field theory (Kugo and Ojima, 1978)

$$N(G, u) = \int d^3x\, \mathbf{N}(\vec{x})^0, \qquad \mathbf{N}(x)^j = \mathbf{G}(x)\mathbf{U}(x)^j_+ \tag{4.16}$$

The gauge-fixing or the Fadeev–Popov field alone gives a Lorentz scalar $\mathbf{GG}$ or a tensor $\mathbf{U}^k_+\mathbf{U}^l_+$.

The subalgebra of the BRS invariants ('gauge invariants') can be generated and spanned by translation eigenvectors only

$$\begin{aligned} &\textbf{eigen } \mathbf{Q}_{+,-}(\mathbb{C}^8) = \{a \in \mathbf{Q}_-(\mathbb{C}^4) \otimes \mathbf{Q}_+(\mathbb{C}^4) | [\![N(\mathrm{G}, \mathrm{u}), a]\!] = 0\} \\ &\text{generated by} \quad \{1, \mathrm{G}, \mathrm{G}^\times, \mathrm{u}, \mathrm{u}^\times, I(\mathrm{B}, \mathrm{G}) + I(\mathrm{a}, \mathrm{u})\} \qquad (4.17) \\ &\langle I(\mathrm{B}, \mathrm{G}) + I(\mathrm{a}, \mathrm{u})\rangle_2 = 0 \end{aligned}$$

The condition of gauge invariance, adequately implemented as BRS invariance for quantum fields, merges with the condition to have only translation eigenstates in the asymptotic state space.

With respect to the Fock form, the subalgebra of the BRS invariants (translation eigenvectors) carries a positive-semidefinite inner product. After factorization with the orthogonal of the Fock form on the BRS-invariant subalgebra (GNS construction), there remains a trivial 'c-number' complex 1-dimensional asymptotic vector space whose basis can be represented by the quantum algebra unit 1.

Nevertheless, the translation representation in the indefinite unitary modality group U(1, 1) is relevant for the interactions, as illustrated by the ordered time integral of the quantization distribution $\mathrm{s}(x|0)$, which has nontrivial contributions from both particle and nonparticle degrees of freedom (Coulomb potential)

$$i \int dx_0\, \epsilon(x_0)[\mathbf{A}(0)^k, \mathbf{A}(x)^j] = \eta^{kj} \frac{\mu}{2\pi|\vec{x}|} \qquad (4.18)$$

If an 'incoming' particle state s, which as a translation eigenstate is simultaneously BRS-invariant, $[\![N, s]\!] = 0$, e.g., with photons $\mathrm{U}^{1,2}$ and other particle representations u^α with modality group U(1), undergoes a time-space development with the translation group generator H, the resulting 'outgoing' state $[H, s]$ remains BRS-invariant, $[\![N, [H, s]]\!] = 0$, since $[H, N] = 0$.

4.3. Unitarization for Heisenberg–Majorana Fields

Heisenberg–Majorana fields realize faithfully space-time translations in the indefinite modality group U(2, 2) with the generator $iQ(\mathrm{b}, \mathrm{g})^j$—formulated in the notation of Section 2.3 without the momenta dependence $\mathrm{b}(\vec{q})$, etc.:

$$\text{in } \mathbf{Q}_+(\mathbb{C}^8): \quad \{\mathrm{b}_\alpha^\times, \mathrm{g}^\beta\} = \{\mathrm{g}_\alpha^\times, \mathrm{b}^\beta\} = \delta_\alpha^\beta \qquad (4.19)$$

$$Q(\mathrm{b}, \mathrm{g})^j = q^j \frac{[\mathrm{b}^\alpha, \mathrm{g}_\alpha^\times] + [\mathrm{g}^\alpha, \mathrm{b}_\alpha^\times]}{2} + \mathrm{g}^\alpha(\rho^j)_\alpha^\beta \mathrm{g}_\beta^\times = q^j I(\mathrm{b}, \mathrm{g}) + N(\mathrm{g})^j$$

g^α, $\mathrm{g}_\alpha^\times$ ('good') are translation eigenvectors, in contrast to the nilvectors b^α, $b_\alpha^\times$ ('bad').

The subalgebra with all time-space translations eigenvectors is characterized by a trivial action for the nilpotent part of the translation representation

$$\textbf{eigen } \mathbf{Q}_+(\mathrm{C}^8) = \{a \in \mathbf{Q}_+(\mathrm{C}^8) \,|\, [N(\mathrm{g})^j, a] = 0\} \quad \text{generated by } \{1, \mathrm{g}^\alpha, \mathrm{g}^\times_\alpha, I(\mathrm{b}, \mathrm{g})\} \tag{4.20}$$

Obviously, the nilpotent part (nilcharge) is compatible with the generators of the time-space translations

$$[Q(\mathrm{b}, \mathrm{g})^j, N(\mathrm{g})^k] = 0 \tag{4.21}$$

In the full field-theoretic formulation the nilcharge $N(\mathrm{g})^j$ is the space integral of the nilcurrent $\mathbf{N}(x)^j$

$$N(\mathrm{g})^j = \int d^3x\, \mathbf{N}(\vec{x})^j, \qquad \mathbf{N}(x)^j = \mathbf{g}(x)^A(\rho^j)^{\dot{A}}_A \mathbf{g}(x)^\times_{\dot{A}} \tag{4.22}$$

The appropriate quantum algebra form for the modality group $\mathbf{U}(2, 2)$ is the indefinite Heisenberg form (Section 3.8)

$$\langle I \rangle_4 = 0 \Rightarrow \begin{cases} \langle [\mathrm{b}^\times_\alpha, \mathrm{g}^\beta] \rangle_4 = \langle [\mathrm{g}^\times_\alpha, \mathrm{b}^\beta] \rangle_4 = 0 \\ \langle \mathrm{b}^\times_\alpha \mathrm{g}^\beta \rangle_4 = \langle \mathrm{g}^\times_\alpha \mathrm{b}^\beta \rangle_4 = \\ \langle \mathrm{g}^\beta \mathrm{b}^\times_\alpha \rangle_4 = \langle \mathrm{b}^\beta \mathrm{g}^\times_\alpha \rangle_4 = \frac{1}{2}\,\delta^\beta_\alpha \end{cases} \tag{4.23}$$

With respect to the indefinite inner product there survives only a trivial complex 1-dimensional asymptotic state space for the Heisenberg–Majorana fields, spanned by the quantum algebra unit 1 (Section 3.7).

The vanishing form for the translations generator $\langle Q(\mathrm{b}, \mathrm{g})^j \rangle = 0$ leads to a trivial expectation function for the quantization opposite commutator of the Heisenberg–Majorana fields

$$\langle [\mathbf{b}(0)^\times_{\dot{A}}, \mathbf{b}(x)^A] \rangle = 0, \qquad \langle [\mathbf{g}(0)^\times_{\dot{A}}, \mathbf{b}(x)^A] \rangle = 0, \qquad \langle [\mathbf{g}(0)^\times_{\dot{A}}, \mathbf{g}(x)^A] \rangle = 0 \tag{4.24}$$

Therewith the Feynman propagators have no spacelike contributions; there are no in- and outgoing particle states (Heisenberg, 1967).

If the causally supported propagator is written as a difference of Feynman and anti-Feynman propagators $[\mathbf{f}(x\,|\,-im) - \mathbf{f}(x\,|\,im)]/2$, e.g.,

$$\langle \mathcal{T}\mathbf{b}(0)^\times \mathbf{b}(x) \rangle = -\epsilon(x_0)\{\mathbf{b}(0)^\times, \mathbf{b}(x)\} = \frac{i}{2\pi} \int \frac{d^4q\; e^{ixq}}{(2\pi)^3} \frac{\rho_k q^k}{(q^2 - m_p^2)^2}$$

$$\frac{1}{q^2 - m_p^2} = \frac{1}{2}\left[\frac{1}{q^2 - m^2 + io} + \frac{1}{q^2 - m^2 - io}\right] \tag{4.25}$$

one may also say that the in- and outgoing states, seen in the integration prescription $\pm io$, compensate each other. Such a compensation is familiar from the 'twin' structure for the (0, 3)-gauge field nonparticle contributions and the two Fadeev–Popov degrees of freedom (Section 4.2).

Although Heisenberg–Majorana fields have no asymptotic spacelike interpretable particle contributions, they can induce nontrivial interactions via their causally supported quantization distributions, e.g., seen in the exponential potential

$$\int dx_0\, \epsilon(x_0)\{\mathbf{b}(0)^\times, \mathbf{b}(x)\} = -2\rho^a\partial_a \int d|x_0|\, \mathbf{s}'(x|m)$$

$$2\int d|x_0|\, \mathbf{s}'(x|m) = -\frac{e^{-|\vec{x}|m}}{\pi m} \tag{4.26}$$

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